The structure of 2-separations of infinite matroids

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Abstract

Generalizing a well known theorem for finite matroids, we prove that for every (infinite) connected matroid M there is a unique tree T such that the nodes of T correspond to minors of M that are either 3-connected or circuits or cocircuits, and the edges of T correspond to certain nested 2-separations of M. These decompositions are invariant under duality.

1 Introduction

A well known theorem of Cunningham and Edmonds [6], proved independently also by Seymour [9], states that for every connected finite matroid M there is a unique tree T such that the nodes of T correspond to minors of M each of which is either 3-connected, a circuit, or a cocircuit, and the edges of T correspond to certain 2-separations of M.

Cunningham and Edmonds also prove that, given such a decomposition tree for M with an assignment of minors and separations of M to its nodes and edges, the same tree with the minors replaced by their duals defines a decomposition tree for the dual of M [6, 8].

Our aim in this paper is to extend these results to infinite matroids, not necessarily finitary. This is less straightforward than the finite case, for two reasons. One is that we have to handle connectivity formally differently, without using rank. A more fundamental difference is that we cannot obtain the desired parts of our decomposition simply by decomposing the matroid recursively, since such a recursion might be transfinite and end with limits beyond our control. Instead, we shall define the parts explicitly, and will then have to show that they do indeed make up the entire matroid and fit together in the desired tree-structure. This is outlined in more detail in Section 2.

Our result has become possible only by the recent axiomatization of infinite matroids with duality [4]. This has already prompted a number of generalizations of standard finite matroid theorems to infinite matroids [1, 2, 3, 5]. The result we prove here appears to be the first such generalization to all matroids, without any assumptions of finitariness, co-finitariness, or a combination of these.

1.1 Connectivity of infinite matroids

A matroid M is *connected* if every two of its elements lie in a common circuit. Higher-order connectivity for finite matroids is usually defined via the rank function, which is not possible for infinite matroids. However, there is a natural rank-free reformulation, as follows.

Consider a partition (X, Y) of the ground set of a matroid M; the sets X and Y may be empty. Given a basis B_X of $M|X$ and a basis B_Y of $M|Y$, the matroid M will be spanned by $B_X \cup B_Y$, so there exists a set $F \subseteq B_X \cup B_Y$ such that $(B_X \cup B_Y) \setminus F$ is a basis of M. Bruhn and Wollan [5] showed that the size $k = |F|$ of this set does not depend on the choices of B_X, B_Y and F, but only on (X, Y). If in addition $|X|, |Y| \geq k+1$, we call (X, Y) a *separation* of M, or more specifically a $(k+1)$-*separation*,[1] or a *separation of order* $k+1$. The matroid M is n-*connected* if it has no ℓ-separation for any $\ell < n$. For M finite, these definitions are equivalent to the traditional ones.

1.2 Tree-decompositions

Let T be a tree. Consider a partition $R = (R_v)_{v \in T}$ of the ground set E of a matroid M into *parts* R_v, one for every node v of T. (We allow $R_v = \emptyset$.) Given an edge $e = vw$ of T, write T_v and T_w for the components of $T - e$ containing v and w, respectively, and put $S(e, v) := \bigcup_{u \in T_v} R_u$ and $S(e, w) := \bigcup_{u \in T_w} R_u$. If each of the partitions $(S(e, v), S(e, w))$ of E, as vw varies over the edges of T, is a separation of M, we call the pair (T, R) a *tree-decomposition* of M. The supremum of the orders of the separations $(S(e, v), S(e, w))$ is the *adhesion* of the decomposition (T, R). If all these separations have the same order k, then we say that (T, R) has *uniform adhesion* k.

Let (T, R) be a tree-decomposition of M of uniform adhesion 2. With every node $v \in T$ we shall associate a matroid M_v, whose ground set will be the set R_v together with some 'virtual elements', one for every edge of T at v. Write F_v for the set of all the edges of T incident with v. As the *circuits* of M_v, we take the sets

$$(C \cap R_v) \cup \{e \in F_v \mid e = vw \text{ with } C \cap S(e, w) \neq \emptyset\}, \tag{1}$$

where C ranges over all the circuits of M not contained in any of the sets $S(vw, w)$. We shall prove in Lemma 4.1 that

$$M_v \text{ is a matroid on } R_v \cup F_v. \tag{2}$$

We call the matroids M_v the *torsos* of the tree-decomposition (T, R).

As we shall see below, if a torso M_v is a circuit of size at least 4 we can partition R_v into two subsets, and correspondingly split v into adjacent nodes v_1, v_2 of T, to obtain another tree-decomposition of M of uniform adhesion 2;

[1] Some authors, including Oxley [8], call this an *exact* $(k+1)$-separation, and use the term '$(k+1)$-separation' for any separation of order at most $k+1$. The tradition of referring to $k+1$, rather than k, as the *order* of a separation with $|F| = k$ may be regrettable but is standard.

in this tree-decomposition, M_{v_1} and M_{v_2} will again be circuits. This split of R_v can be done in more than one way. Hence even if we aim to make the sets R_v as small as possible, our tree-decomposition of M of uniform adhesion 2 will not in general be unique.

To achieve uniqueness, we therefore forbid 'adjacent' cycles and cocyles, as follows. Call a tree-decomposition (T, R) *irredundant* if

(i) all torsos have size at least three; and

(ii) for every edge vw of T, the torsos M_v, M_w are not both circuits and not both cocircuits.

1.3 Statement of results

The following infinite decomposition theorem is our main result:

Theorem 1.1.

(i) *Every connected matroid with at least three elements, finite or infinite, has an irredundant tree-decomposition of uniform adhesion 2 every torso of which is either* 3*-connected, a circuit, or a cocircuit.*

(ii) *This decomposition is unique in the sense that for any two such tree-decompositions, (T, R) and (T', R') say, there is an isomorphism $v \mapsto v'$ between the trees such that $R_v = R'_{v'}$ for all $v \in T$.*

Since k-separations of a matroid M are also k-separations of its dual M^* [5], a tree-decomposition of M is also one of M^*, with the same adhesion. Moreover, the torsos corresponding to a given tree node are duals of each other:

Theorem 1.2. *Every tree-decomposition (T, R) of a connected matroid M is also a tree-decomposition of its dual M^*. If (T, R) has uniform adhesion 2 for M, it has uniform adhesion 2 also for M^*, and $(M_v)^* = (M^*)_v$ for all $v \in T$. In particular, M and M^* have the same unique irredundant tree-decomposition.*

The notation we use in this paper is as follows. Axiom systems for infinite matroids can be found in [4]. For other terminology we follow Oxley [8], or [7] for graphs. The letter M always denotes a matroid. Its ground set, set of bases, and set of circuits will be denoted by $E(M)$, $\mathcal{B}(M)$ and $\mathcal{C}(M)$, respectively. Given $S \subseteq E(M)$, we let $M|S$ and M/S denote the restriction of M to S and the contraction of S in M, respectively, and write $S^{\complement} := E(M) \setminus S$. The dual matroid of M is denoted by M^*.

2 Definitions, and outline of proof

In this section we give an outline of our proof of Theorem 1.1, which is described in detail in the rest of this paper. In particular, we describe the construction of the tree-decomposition whose existence is claimed in the theorem, and introduce

the concepts needed to define it. We do not assume familiarity with the standard finite proof of Cunningham and Edmonds [6], but for readers familiar with that proof we emphasize the points where our potentially infinite setting requires a different approach. Throughout this section, let M be a fixed connected matroid.

2.1 A tree of 2-separations

Two k-separations $(A, A^\complement)$ and $(B, B^\complement)$ of M are said to be *nested* if one of the four sets $A, A^\complement, B, B^\complement$ contains another. As one easily checks, this is equivalent to saying that at least one of the four sets $A \cap B$, $A^\complement \cap B$, $B^\complement \cap A$, $A^\complement \cap B^\complement$ is empty. Two separations that are not nested are said to *cross*. A *good k-separation* is one that is nested with all other k-separations.

When (T, R) is a tree-decomposition of M then the partitions $(S(e,v), S(e,w))$ of $E(M)$ that correspond to the edges vw of T are pairwise nested: this is because the corresponding pairs (T_v, T_w) of subtrees of T are nested, a property of trees that is easily checked. Hence in order to construct any tree-decomposition of M we shall have to pick from the set of all 2-separations some suitable nested subset. We shall show that the set of all good 2-separations, which is obviously nested, gives rise to the desired tree-decomposition for Theorem 1.1.

For infinite matroids, this is not entirely trivial. One difficulty is that a decreasing chain $(A, A^\complement), \dots, (B, B^\complement)$ of separations, one where $A \supsetneq \dots \supsetneq B$, can now be infinite. If our claim that the good 2-separations correspond to the edges of a decomposition tree is true, then such infinite chains must not occur within the set of good 2-separations. For if $(A, A^\complement)$ and $(B, B^\complement)$ correspond to tree edges, then the tree will have only finitely many edge between these two, and hence the corresponding finite set of good 2-separations must be the only good 2-separations $(C, C^\complement)$ satisfying $A \supsetneq C \supsetneq B$ or $A \supsetneq C^\complement \supsetneq B$.

Since $B \neq \emptyset$, as $(B, B^\complement)$ is a 2-separation, the following lemma from Section 5 implies that there are indeed no such infinite chains of good 2-separations:

Proposition 2.1. *Let $S_1 \supsetneq S_2 \supsetneq \dots$ be an infinite sequence of subsets of $E(M)$ such that every partition $(S_i, S_i^\complement)$ is a good 2-separation of M. Then $\bigcap_{i=1}^\infty S_i = \emptyset$.*

Another new difficulty in turning the set of good 2-separations into a tree-decomposition is to define the parts corresponding to the nodes of the tree, indeed to define the tree itself. For M finite, Cunningham and Edmonds obtain these parts and their torsos simultaneously, by splitting M recursively along good 2-separations of the 'current' matroid (not of M) and adding a virtual element to each side in every split. When the recursion stops, the 'current' matroids are the desired torsos. When M is infinite, such a recursion would have to be transfinite, and it is not clear how M should induce matroids on the parts of the partitions (plus some virtual elements) that arise at limit steps. We shall therefore define those matroids, the torsos of our tree-decomposition, explicitly.

2.2 Constructing the tree-decomposition

In Section 7 we therefore define the decomposition tree explicitly, as follows. With any *symmetrical* nested set $\mathcal{F}$ of 2-separations of M, one such that $(A, A^{\complement}) \in \mathcal{F}$ implies $(A^{\complement}, A) \in \mathcal{F}$, we shall associate a tree $T = T_{\mathcal{F}}$. (In the intended application, $\mathcal{F}$ will be the set of good 2-separations, and T will be our decomposition tree.) Let us define the edges of T first, and then its vertices, or *nodes*. To define the edges, consider the partial ordering on $\mathcal{F}$ given by writing $(A, A^{\complement}) \leq (B, B^{\complement})$ whenever $A \subseteq B$. As the *edges* of T we take the 2-separations in $\mathcal{F}$ up to inversion:

$$E(T_{\mathcal{F}}) := \big\{\{(A, A^{\complement}), (A^{\complement}, A)\} : (A, A^{\complement}) \in \mathcal{F}\big\}. \tag{3}$$

To define the nodes of T, we call $(A, A^{\complement})$ and $(B, B^{\complement})$ *equivalent* if either $(A, A^{\complement}) = (B, B^{\complement})$ or $(A^{\complement}, A)$ is a predecessor of $(B, B^{\complement})$ in this ordering, i.e., if $A^{\complement} \subset B$ but there is no $(C, C^{\complement}) \in \mathcal{F}$ such that $A^{\complement} \subset C \subset B$. This is indeed an equivalence relation, and we take its classes as the nodes of T:

$$V(T_{\mathcal{F}}) := \big\{[(A, A^{\complement})] : (A, A^{\complement}) \in \mathcal{F}\big\}. \tag{4}$$

We then let the edge $\{(A, A^{\complement}), (A^{\complement}, A)\}$ join the nodes $[(A, A^{\complement})]$ and $[(A^{\complement}, A)]$; these are distinct classes, since $(A, A^{\complement})$ is not equivalent to $(A^{\complement}, A)$. Note that the degree of a node v in T is simply its cardinality, the number of good 2-separations in the equivalence class v.

In order to turn T into a decomposition tree of M, we have to associate with every node v of T a part $R_v \subseteq E(M)$ of the intended tree-decomposition (T, R), where $R = (R_v)_{v \in T}$. We do this by setting

$$R_v := \bigcap \{\, A \mid (A, A^{\complement}) \in v \,\}\,. \tag{5}$$

When v has degree at least 3 in T, i.e. if $|v| \geq 3$, this set R_v can be empty. We shall have to prove both that the graph T thus defined is acyclic and that it is connected. Connectedness will follow from Lemma 2.1.

Our aim will be to prove the following.

Lemma 2.2. *When $\mathcal{F}$ is the set of all good 2-separations of M, then $(T_{\mathcal{F}}, R)$, as defined above, is a tree-decomposition of M witnessing Theorem 1.1 (i).*

2.3 Characterizing the torsos

From the tree-decomposition (T, R) and its parts R_v we define the *torsos* M_v as in (1). These torsos will be studied in detail in Section 4. Our aim then is to prove that they are 3-connected matroids, or circuits, or cocircuits. This will be done in Sections 7 and 6, in two steps.

The first step will be to show that these torsos have no good 2-separations. Or equivalently, that any good 2-separation of a torso M_v would give rise to a

good 2-separation of M that splits R_v, which by definition of R_v does not exist. This will be done in Section 7. We also show there that our tree-decomposition is irredundant. These properties, together with the remark following Lemma 2.3 below, already imply its uniqueness as claimed in Theorem 1.1 (ii).

As the second step, in Section 6, we show that the property of our torsos just established (that they have no good 2-separations) implies that they are 3-connected, circuits, or cocircuits:

Lemma 2.3. *If M has no good 2-separation, it is 3-connected, a circuit, or a cocircuit.*

The converse of this is easy: 3-connected matroids have no 2-separations at all, and any 2-separation of a circuit or cocircuit crosses another 2-separation.

Lemma 2.3 is in turn proved in two steps; these are captured by the following two lemmas (which imply Lemma 2.3).

Lemma 2.4. *If M has no good 2-separation but is not 3-connected, then for every two elements x, y the partition $(\{x, y\}, \{x, y\}^{\complement})$ is a 2-separation.*

Lemma 2.5. *If M is such that for every two elements x, y the partition $(\{x, y\}, \{x, y\}^{\complement})$ is a 2-separation, then M is a circuit or a cocircuit.*

The converse of Lemma 2.5 is again easy.

Lemmas 2.4 and 2.5 are proved in [6] for finite matroids, but the proofs do not adapt to infinite matroids. In Section 6 we provide alternative proofs.

3 Properties of 2-separations

The purpose of this section is to study the properties of 2-separations in infinite matroids. This is necessary, since the standard proofs for finite matroids [8] do not always carry over.

Of the various axiom systems for infinite matroids established in [4] we shall use the *circuit axioms*:

(C1) The empty set is not a circuit.

(C2) No circuit is a proper subset of another.

(C3) Whenever $X \subseteq C \in \mathcal{C}(M)$ and $\{C_x : x \in X\}$ is a family of circuits such that $x \in C_y \iff x = y$ for all $x, y \in X$, then for every $z \in C \setminus (\bigcup_{x \in X} C_x)$ there exists a circuit C' satisfying $z \in C' \subseteq (C \cup (\bigcup_{x \in X} C_x)) \setminus X$.

(CM) For every independent set I (those sets not contained in any circuit C) and any set S containing I, there is a maximal independent subset of S containing I.

Axiom (C3) generalizes the traditional finite circuit elimination axiom, and is referred to as the *infinite circuit elimination axiom.* The (CM) axiom is redundant for finite matroids.

The following notation will be frequently used. If B is a base of a matroid M and $e \in E(M) \setminus B$, then we write $C_M(e, B)$ to denote the fundamental circuit of e into B; if the matroid M is understood, then we omit the subscript.

3.1 New 2-separations from crossing 2-separations

Two crossing k-separations define four nonempty disjoint sets, by definition, to which we refer as the *quadrants* of these two crossing separations. In the next two lemmas, we shall see that for $k = 2$, certain unions of these quadrants give rise to other 2-separations.

Lemma 3.2 below asserts that if a quadrant of two crossing 2-separations and its complement both have size at least 2, then they form a 2-separation as well. The following claim facilitates in our proof of Lemma 3.2.

Claim 3.1. *Let $(S_1, S_1^{\complement})$ and $(S_2, S_2^{\complement})$ be two crossing 2-separations of a connected matroid M. If*

- *$S_1 \cap S_2$ and $(S_1 \cap S_2)^{\complement}$ both have size at least 2, and*
- *$\left(S_1 \cap S_2, (S_1 \cap S_2)^{\complement}\right)$ is not a 2-separation of M,*

then $|(S_1 \cup S_2)^{\complement}| \geq 2$.

Proof. Suppose not, then $|(S_1 \cup S_2)^{\complement}| = 1$ as $(S_1, S_1^{\complement})$ and $(S_2, S_2^{\complement})$ are crossing; put $(S_1 \cup S_2)^{\complement} = \{x\}$. Let B be a base of $M|(S_1 \cap S_2)$, and let B_M be a base of M containing B and not containing x; such a base exists as $E(M) - x$ is spans M since M is connected. Put $I = B_M \cap (S_1^{\complement} \cap S_2)$ and $I' = B_M \cap (S_1 \cap S_2^{\complement})$. Since both $(S_1, S_1^{\complement})$ and $(S_2, S_2^{\complement})$ are 2-separations of M, we may assume, without loss of generality, that $B + I$ is a base of S_2. Also, by assumption, $\left(S_1 \cap S_2, (S_1 \cap S_2)^{\complement}\right)$ is not a 2-separation of M, so there exist two elements e and e' in $(S_1 \cap S_2)^{\complement}$ such that $I + I' + \{e, e'\}$ is independent.

As $(S_1, S_1^{\complement})$ is a 2-separation, either $B + I'$ is a base of S_1 or I is a base of $S_1^{\complement}$. In either case, $B' = I + I' + \{e + e'\}$ is a base of $(S_1 \cap S_2)^{\complement}$, such that each of $B' \cap (S_1^{\complement} \cap S_2)$ and $B' \cap (S_2^{\complement} \cap S_1)$ span x. Two circuits containing x witnessing this spanning of x by both these sets yield a circuit in B', by the circuit elimination axiom. This is a contradiction as B' is independent. □

A function $f : E(M) \to \mathbb{R}$ is called *submodular* if

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \text{ for all } X, Y \subset E(M). \tag{6}$$

As mentioned in the Introduction, Bruhn and Wollan [5] gave a rank free definition for the connectivity of a matroid. Given a matroid M and two independent sets I and J of M, we follow [5] in defining

$$\operatorname{del}(I, J) = \min\{|F| : F \subseteq I \cup J,\ (I \cup J) \setminus F \in \mathcal{I}(M)\}.$$

The *connectivity function* φ of M is now defined as follows. Given $X \subseteq E(M)$, let B_X and $B_{X^{\complement}}$ be two arbitrary bases of $M|X$ and $M|X^{\complement}$, respectively. Set

$$\varphi(X) = \mathrm{del}(B_X, B_{X^{\complement}}).$$

The function φ is well defined [5, Lemma 14] and submodular [5, Lemma 19].

Lemma 3.2. (Corner lemma)
Let $(S_1, S_1^{\complement})$ and $(S_2, S_2^{\complement})$ be two crossing 2-separations of a connected matroid M such that $S_1 \cap S_2$ and $(S_1 \cap S_2)^{\complement}$ both have size at least 2. Then, $(S_1 \cap S_2, (S_1 \cap S_2)^{\complement})$ is a 2-separation.

Proof. By assumption, $\varphi(S_1) = \varphi(S_2) = 2$. Then, submodularity of φ and the assumption that M is connected yield

$$2 \leq \varphi(S_1 \cap S_2) \leq 4 - \varphi(S_1 \cup S_2).$$

As M is connected and $|(S_1 \cup S_2)^{\complement}| \geq 2$, by Claim 3.1, we have $\varphi(S_1 \cup S_2) \geq 2$ and the lemma follows. □

The next lemma asserts that the union of two "opposing" quadrants of two crossing 2-separations and the complement of such a union form a 2-separation as well.

Lemma 3.3. (Symmetric difference lemma)
If $(S_1, S_1^{\complement})$ and $(S_2, S_2^{\complement})$ are two crossing 2-separations of M, then $(S_1 \Delta S_2, (S_1 \Delta S_2)^{\complement})$ is a 2-separation of M.

Proof. Put $X_1 = S_1 \cap S_2, X_2 = S_1^{\complement} \cap S_2, X_3 = S_1^{\complement} \cap S_2^{\complement}$, and $X_4 = S_1 \cap S_2^{\complement}$. As S_1 and S_2 cross these are all non-empty. Let $B_{S_1 \Delta S_2} \in \mathcal{B}(M|(X_1 \cup X_3))$ and choose $B_M \in \mathcal{B}(M)$ satisfying $B_{S_1 \Delta S_2} \subseteq B_M$. Put $B_i = B_M \cap X_i$, for $1 \leq i \leq 4$.

Assume, towards a contradiction, that $(X_1 \cup X_3, (X_1 \cup X_3)^{\complement})$ is not a 2-separation of M so that $B_2 \cup B_4$ is missing at least two elements from being a base of $M|(X_2 \cup X_4)$. Let $e_2, e_4 \in X_2 \cup X_4$ be two such elements. We may assume that

$$e_4 \in X_4 \text{ and } e_2 \in X_2. \tag{7}$$

Indeed, if $e_2, e_4 \in X_4$ (equivalently X_2), then $(X_4, (X_4)^{\complement})$ is not a 2-separation of M in contradiction to the corner lemma. To see this, extend $B_{S_1 \Delta S_2} \cup B_2$ into a base of $M|(X_1 \cup X_2 \cup X_3)$ and $B_4 \cup \{e_2, e_4\}$ into a base of $M|X_4$; from the union of which at least two elements must be removed in order to obtain a base of M.

Consider e_2. Since $B_2 \cup B_4 \cup \{e_2, e_4\}$ is independent, $B_2 + e_2$ is independent. In addition, at least one of the sets $(B_2 + e_2) \cup B_1$ and $(B_2 + e_2) \cup B_3$ is independent for otherwise e_2 has two distinct fundamental circuits into B_M. As a similar argument holds for e_4 and B_4, we may put more generally that

$$\text{for } j \in \{2, 4\},\ B_i \cup (B_j + e_j) \text{ is independent for at least one } i \in \{1, 3\}. \tag{8}$$

In fact,

$$\text{there is an } i \in \{1,3\} \text{ such that } B_i \cup (B_j + e_j) \text{ is dependent for } j \in \{2,4\}. \quad (9)$$

To see (9), suppose, without loss of generality, that $B_1 \cup (B_4 + e_2)$ and $B_3 \cup (B_4 + e_4)$ are independent Choose, now, $B_{1,2} \in \mathcal{B}(M|(X_1 + X_2))$ and $B_{3,4} \in \mathcal{B}(M|(X_3 + X_4))$ satisfying $B_1 \cup (B_2 + e_2) \subseteq B_{1,2}$ and $B_3 \cup (B_4 + e_4) \subseteq B_{3,4}$, respectively. From $B_{1,2} \cup B_{3,4}$, at least two elements must be removed in order to obtain a base of M; a contradiction to $(S_2, S_2^{\complement})$ being a 2-separation.

Suppose then, without loss of generality, that $B_1 + e_1$ and $B_1 + e_2$ are dependent sets, by (9); so that $C(B_M, e_1) \subseteq B_1 + e_1$ and $C(B_M, e_2) \subseteq B_1 + e_2$. Consequently, $B_2 \cup B_3 \cup B_4 \cup \{e_1, e_2\}$ is independent. We may assume that $|X_1| \geq 2$; for if $X_1 = \{x\}$, then e_1 and e_2 are parallel, by applying circuit elimination on $C(B_M, e_1), C(B_M, e_1)$, and x. Choose bases $B_1' \in \mathcal{B}(M|X_1)$ and $B_{2,3,4} \in \mathcal{B}(M|(X_2 \cup X_3 \cup X_4))$ satisfying $B_1 \subseteq B_1'$ and $B_2 \cup B_3 \cup B_4 \cup \{e_1, e_2\} \subseteq B_{2,3,4}$, respectively. Then, $B_1' \cup B_{2,3,4}$ indicates that $(X_1, (X_1)^{\complement})$ is not a 2-separation of M, contradicting the corner lemma.

Hence $(S_1 \Delta S_2, (S_1 \Delta S_2)^{\complement})$ is a 2-separation of M. Moreover as X_1, X_2, X_3, X_4 are all nonempty, $|S_1 \Delta S_2|, |(S_1 \Delta S_2)^{\complement}| \geq 2$ and the separation is proper as desired. □

3.2 The limit of infinitely many nested k-separations

In this section, we consider infinite sequences of nested k-separations. In particular, our next lemma asserts that the limit of a nested sequence of k-separations is again an ℓ-separation for some $\ell \leq k$, or a degenerate partition that cannot be a k-separation because one side is too small. This follows from [5, Lemma 20]; nevertheless, we include here a short proof for the convenience of the reader.

Lemma 3.4. *Let $k \geq 1$ be an integer, and $\mathcal{S}$ an $\subseteq$-chain of subsets of $E(M)$ such that $(S, S^{\complement})$ is a k-separation of M for every $S \in \mathcal{S}$. Then either $\left|\bigcap \mathcal{S}\right| < k$, or $(\bigcap \mathcal{S}, (\bigcap \mathcal{S})^{\complement})$ is an ℓ-separation of M for some integer $\ell \leq k$.*

Proof. Let $S_\cap := \bigcap \mathcal{S}$. Pick a basis B of $M|S_\cap$, extend it to a basis B_M of M, and extend $B_M \cap (S_\cap)^{\complement}$ to a basis B' of $M|(S_\cap)^{\complement}$. If $B' \setminus B_M$ has size $< k$ then $(S_\cap, (S_\cap)^{\complement})$ is as desired. We may thus assume that $B' \setminus B_M$ contains a k-set Y.

Since $Y \subseteq (S_\cap)^{\complement}$ is finite, there exists an $S \in \mathcal{S}$ such that $Y \subseteq S^{\complement}$. Extend $B_M \cap S$ to a basis B_S of $M|S$, and $(B_M \cap S^{\complement}) \cup Y$ $(\subseteq B')$ to a basis $B_{S^{\complement}}$ of $M|S^{\complement}$. Then $B_S \cup B_{S^{\complement}}$ exceeds its subset B_M by at least the k-set Y, contrary to our assumption that $(S, S^{\complement})$ is a k-separation. □

It would be interesting to know whether Lemma 3.4 always holds with $\ell = k$.

3.3 2-sums of infinite matroids

In this section, we consider the operation of taking a *2-sum* of two matroids. In the sequel, we shall use this operation to separate a connected matroid along

a given 2-separation into two matroids; each a minor of the original matroid. The 2-sum operation, its properties, and typical uses are well known for finite matroids (see e.g., [8]); nevertheless, our infinite setting mandates that we study this operation and provide alternative proofs to some of its properties in a manner suitable for infinite matroids.

Let M_1 and M_2 be two matroids having a single element e in common, that is, $E(M_1) \cap E(M_2) = \{e\}$. Let $\mathcal{C}_e$ denote the set comprised of the circuits of $M_i, i = 1, 2$ not containing e together with the sets of the form $(C_1 - e) \cup (C_2 - e)$, whenever $e \in C_1 \cap C_2$ and $C_1 \in \mathcal{C}(M_1)$, and $C_2 \in \mathcal{C}(M_2)$. The set system $\mathcal{C}_e$ then defines a matroid as follows.

Lemma 3.5. *The set system $\mathcal{C}_e$ is the set of circuits of a matroid whose ground set is $E(M_1) \cup E(M_2) - e$.*

The matroid defined in Lemma 3.5 is called the 2-*sum* of M_1 and M_2, and is denoted by $M_1 \oplus_2 M_2$. In what follows we prove Lemma 3.5 in a manner suitable for infinite matroids. To that end, we prove the following.

Lemma 3.6. *If $(S, S^{\complement})$ is a 2-separation of M, then there exist two matroids M_1, M_2 such that $E(M_1) = S + e$ and $E(M_2) = S^{\complement} + e$, where $e \notin E(M)$ so that $M = M_1 \oplus_2 M_2$. Moreover, M_i is isomorphic to a minor of M, for $i = 1, 2$.*

Lemma 3.6 is a corollary of Lemma 4.1 (stated below). The latter is one of the main results of Section 4. In the remainder of this section we establish what we call the *infinite switching lemma*, which is stated in Lemma 3.9; this lemma will be used repeatedly throughout and in particular in the proof of Lemma 4.1. For future reference, it will be convenient for us to mention a special case of the infinite switching lemma, to which we refer simply as the *switching lemma*; the latter appears in [8] and here it is stated in Lemma 3.8. One should note that the proof found in [8] for the switching lemma does not fit for infinite matroids.

A circuit C of M is said to *cross* a 2-separation $(S, S^{\complement})$ of M if C meets both S and $S^{\complement}$. The following lemma is that of [8] and its proof is included here for completeness.

Lemma 3.7. *If C_1 and C_2 are circuits of M crossing a 2-separation $(S, S^{\complement})$ of M, then $C_1 \cap S$ is not a proper subset of $C_2 \cap S$.*

Proof. Assume, to the contrary, that $C_1 \cap S \subsetneq C_2 \cap S$ and let $e_1 \in C_1 \cap S$ and $e_2 \in (C_2 \setminus C_1) \cap S$. Choose $B_S \in \mathcal{B}(M|S)$ and $B_{S^{\complement}} \in \mathcal{B}(M|S^{\complement})$ satisfying $C_1 \cap S \subseteq B_S$ and $C_1 \cap S^{\complement} \subseteq B_{S^{\complement}}$, respectively. Since S is a 2-separation, $Z = (B_S \cup B_{S^{\complement}}) \setminus \{e_1, e_2\}$ is independent. Observe now that Z is spanning; indeed, $E(M) - e_1$ is spanned by Z, and as $C_1 - e_1 \subseteq Z$, the element e_1 is spanned by Z as well. This contradicts the assumption that S is a 2-separation. $\square$

As mentioned above, a point to notice about the next lemma is that in order to have it hold for infinite matroids, one seems to need the infinite circuit elimination axiom, i.e., (C3).

Lemma 3.8. (Switching lemma)
If C_1 and C_2 are circuits of M crossing a 2-separation $(S, S^{\complement})$ of M, then $(C_1 \cap S) \cup (C_2 \cap S^{\complement})$ is a circuit.

Our interest in the switching lemma exceeds the need of using it in order to prove Lemma 3.6; indeed, in the sequel we shall make frequent use of a more general switching lemma stated next, and to which we refer as the *infinite switching lemma*. Lemma 3.8 is a special case of the infinite switching lemma.

Lemma 3.9. (Infinite switching lemma)
Let $\{S_i : i \in I\}$ be a set of disjoint subsets of $E(M)$ where $(S_i, S_i^{\complement})$ is a 2-separation of M for every $i \in I$. If

(1) *C_1 and C_2 are circuits each crossing $(S_i, S_i^{\complement})$ for all i, and*

(2) *C_2 meets $\left(\bigcup_{i \in I} S_i\right)^{\complement}$ if C_1 does,*

then $(C_1 \cap \bigcup_{i \in I} S_i) \cup (C_2 \cap (\bigcup_{i \in I} S_i)^{\complement})$ is a circuit.

Proof. Put $C = (C_1 \cap \bigcup_{i \in I} S_i) \cup (C_2 \cap (\bigcup_{i \in I} S_i)^{\complement})$. Suppose, first, that C is independent and extend C into a base B_M of M. Set $X_i = B_M \cap S_i$. Either there exists an i such that $B_M \cap S_i^{\complement}$ is a base of $M|S_i^{\complement}$, or there is no such i. In the former case, let z be an element of X_i, set $V = C_1 \setminus B_M$, and for each $e \in V$ let C_e denote its fundamental circuit into $B_M \cap S_i^{\complement}$. Then, by the infinite circuit elimination axiom applied to C_1, z, V, and $\{C_e : e \in V\}$, there exists a circuit in $C_1 \cup \bigcup_{e \in V} C_e \setminus V \subseteq B_M$; a contradiction.

We may now assume that the latter case holds; that is, $B_M \cap S_i^{\complement}$ is not a base of $M|S_i^{\complement}$ for any i. This together with the assumption that $(S_i, S_i^{\complement})$ is a 2-separation of M for every i imply that

$$X_i = B_M \cap S_i \text{ is a base of } M|S_i \text{ for every } i. \tag{10}$$

We arrive at a contradiction in this case as follows.

As C_2 is not contained in B, we may assume, without loss of generality, that

$$Y = (C_2 \cap S_1) \setminus X_1 \text{ is nonempty.} \tag{11}$$

Set

$$V = C_2 \setminus (B \cup Y), \tag{12}$$

and note that $V \subseteq \bigcup_{i>1} (C_2 \cap S_i) \setminus X_i$. We may assume that

$$V \text{ is nonempty.} \tag{13}$$

Indeed, for otherwise, choose a $y \in C_2 \cap X_i$ for some $i \neq 1$, such an element y exists as C_2 crosses $(S_1, S_1^{\complement})$. Applying the infinite circuit elimination axiom to C_2, y, Y, and $\{C_{M|S_i}(e, X_i) : e \in Y\}$, yields a circuit contained in B_M which is a contradiction.

For each $e \in V$, there exists an i_e such that $e \in S_{i_e}$. Let C_e denote the fundamental circuit of e into X_{i_e} in $M|S_{i_e}$. In addition, choose an element $z \in Y$. Then, the infinite circuit elimination axiom applied to C_2, z, V, and $\{C_e : e \in V\}$ yields that there exists a circuit C_3 contained in $\big(C_2 \cup \bigcup_{e \in V} C_e\big) \setminus V$ so that

$$C_3 \setminus Y \subseteq B_M. \tag{14}$$

Observe that if C_3 does not meet $\big(\bigcup_{e \in V} C_e\big) \setminus V$, then $C_3 \subset C_2$ which is a contradiction. Consequently,

$$C_3 \text{ crosses } (S_1, S_1^{\complement}). \tag{15}$$

Then, the infinite circuit elimination axiom applied to C_3, an element of $C_3 \cap S_1^{\complement}$, the set Y, and the set $\{C_{M|S_1}(e, X_1) : e \in Y\}$, yields a circuit contained in B_M which is a contradiction.

Suppose, second, that C is dependent, and let C_3 be a circuit contained in C. We show that C coincides with C_3. As C_3 is not properly contained in neither C_1 nor C_2 it follows that C_3 meets $C_2 \cap (\bigcup_i S_i)^{\complement}$ and also meets $C_1 \cap S_i$ for at least one $i \in I$. In particular, there exists an $i \in I$ such that C_3 crosses $(S_i, S_i^{\complement})$.

Let $I' \subseteq I$ be those indices $i \in I$ such that C_3 crosses $(S_i, S_i^{\complement})$. By Lemma 3.7, $C_3 \cap S_i = C_1 \cap S_i$ for each $i \in I'$. Next, consider $C' = (C_2 \cap \bigcup_{i \in I'} S_i) \cup (C_3 \cap (\bigcup_{i \in I'} S_i)^{\complement})$. If C' is a proper subset of C_2 and thus independent, then we are in the previous case with I replaced with I'. The set C' is not a proper subset of C_2 provided that

$$C_3 \cap \Big(\bigcup_{i \in I'} S_i\Big)^{\complement} = C_2 \cap \Big(\bigcup_{i \in I'} S_i\Big)^{\complement}. \tag{16}$$

This has two implications. First, it holds that $C_3 \cap \big(\bigcup_{i \in I} S_i\big)^{\complement} = C_2 \cap \big(\bigcup_{i \in I} S_i\big)^{\complement}$. Second, it implies that $I' = I$. Indeed, if $I' \subset I$, then (16) implies that $C_3 \cap S_i = C_2 \cap S_i$ for each $i \in I \setminus I'$ implying that C_3 does cross $(S_i, S_i^{\complement})$ for an $i \in I \setminus I'$ which is a contradiction to the definition of I'.

To summarize this case, we have just shown that C_3 coincides with C and the lemma follows. □

4 Localizations

In this section, we study a notion to which we refer as a *localization*; this is essentially a minor of a connected matroid M that has been "isolated" or "pointed at" by a certain set of 2-separations. In particular, torsos (as defined in the previous sections) are localizations with the "localizing" 2-separations chosen all to be good.

Throughout this section, $\mathcal{U} = \{X_i : i \in I\}$ is a set of disjoint subsets of a connected matroid M where $(X_i, X_i^{\complement})$ is a 2-separation of M for all i. Roughly speaking, a localization will be a matroid obtained by essentially contracting

M onto the complement of $\bigcup X_i$, and then adding certain "virtual" elements instead of the members of $\mathcal{U}$. To prove that the resulting object is, in fact, a matroid, we will show that the set comprised of circuits of M that are not contained in any member of $\mathcal{U}$ gives rise to a set system that in turn defines the set of circuits of matroid. We now make this precise.

Let us write

$$R(\mathcal{U}) = E(M) \setminus \bigcup_{i \in I} X_i \tag{17}$$

to denote the elements of M not contained in any member of U. These elements are called the *real* elements of the intended matroid. The ground set of the intended matroid is given by

$$E(\mathcal{U}) = \{e_i : X_i \in \mathcal{U}\} \cup R(\mathcal{U}), \tag{18}$$

where the elements e_i are distinct and all are not in $E(M)$; we call these elements *virtual*. Next, given a subset $Y \subseteq E(MN)$, we set

$$\varphi_{\mathcal{U}}(Y) = \{e_i : Y \cap X_i \neq \emptyset\} \cup (Y \cap R(\mathcal{U}))\,, \tag{19}$$

and say that Y *induces* $\varphi_{\mathcal{U}}(Y)$. So $\varphi_{\mathcal{U}}$ is simply a mapping from the subsets of $E(M)$ to the subsets of $E(\mathcal{U})$ Finally, let $\mathcal{C}_{\mathcal{U}}(M)$ denote the circuits of M not contained in any X_i, that is,

$$\mathcal{C}_{\mathcal{U}}(M) = \{C \in \mathcal{C}(M) | \nexists i \text{ such that } C \subseteq X_i\}. \tag{20}$$

The following is the first main result of this section.

Lemma 4.1. *The set $\mathcal{C}(\mathcal{U}) = \{\varphi_{\mathcal{U}}(C) : C \in \mathcal{C}_{\mathcal{U}}(M)\}$ is the set of circuits of a matroid whose ground set is $E(\mathcal{U})$.*

Definition 4.2. *The matroid of Lemma 4.1 is called the* localization of M at $\mathcal{U}$*, and is denoted by $M_{\mathcal{U}}$.*

The second main result of this section is Lemma 4.10. This lemma essentially asserts that a good 2-separation of a localization $M_{\mathcal{U}}$ of M gives rise to a good 2-separation of M. We shall use this lemma in Section 7 to argue that torsos admit no good 2-separations. We postpone discussion of this lemma until later sections.

This section is organized as follows. Sections 4.1 and 4.2 are dedicated to the proof of Lemma 4.1. In these sections we verify that $\mathcal{C}(\mathcal{U})$ satisfies the circuit axioms and thus defines a matroid. In Section 4.3, we prove Lemma 4.10.

4.1 The axioms (C1)–(C3) for localizations

We begin by verifying that $\mathcal{C}(\mathcal{U})$ satisfies (C1). To see this note that the empty set is not in $\mathcal{C}(M)$, by (C1), and that the image of a nonempty set under $\varphi_{\mathcal{U}}$ is a nonempty set. As a result we have the following.

Claim 4.3. *The empty set is not in $\mathcal{C}(\mathcal{U})$ so that $\mathcal{C}(\mathcal{U})$ satisfies (C1).*

To verify (C2), we observe the following.

Claim 4.4. *If $C_1, C_2 \in \mathcal{C}(\mathcal{U})$, then C_1 is not a proper subset of C_2.*

Proof. Suppose that C_1 is a proper subset of C_2, and let $C_1', C_2' \in \mathcal{C}_{\mathcal{U}}(M)$ be circuits satisfying $C_1 = \varphi(C_1')$ and $C_2 = \varphi(C_2')$. By the infinite switching lemma (see Lemma 3.9), we may assume that $C_1' \cap X_i = C_2' \cap X_i$ for all i where $C_1' \cap X_i \neq \emptyset$. It follows now that C_1' is a proper subset of C_2'; a contradiction to axiom (C2) for M. $\square$

Next, we consider the axiom (C3).

Claim 4.5. *$\mathcal{C}(\mathcal{U})$ satisfies the infinite circuit elimination axiom (C3).*

Proof. Let $C_{\mathcal{U}} \in \mathcal{C}(\mathcal{U})$, let $z_{\mathcal{U}} \in C_{\mathcal{U}}$, and let $V_{\mathcal{U}} \subseteq C_{\mathcal{U}}$ such that $z_{\mathcal{U}} \notin V_{\mathcal{U}}$. Suppose now that $\{C_v : v \in V_{\mathcal{U}}\}$ is a subset of $\mathcal{C}(\mathcal{U})$ satisfying the property stated in axiom (C3) that $u \in C_v \iff u = v$ for all $u, v \in V_{\mathcal{U}}$. We prove that there is a member of $\mathcal{C}(\mathcal{U})$ contained in $\left(C_U \cup \bigcup_{v \in V_{\mathcal{U}}} C_v\right) \setminus V_{\mathcal{U}}$; moreover, member of $\mathcal{C}(\mathcal{U})$ contains $z_{\mathcal{U}}$.

Let C_M be a circuit of M satisfying $C_{\mathcal{U}} = \varphi(C_M)$. For an X_i, for which the element $e_i \in C_{\mathcal{U}}$, let $D_i = C_M \cap X_i$, and let $d_i \in D_i$. For a $v \in V_{\mathcal{U}}$, let C_v' be a circuit of M satisfying $C_v = \varphi(C_v')$. By the infinite switching lemma (see Lemma 3.9), we may assume that, for all $v \in V_{\mathcal{U}}$ and $i \in I$: $e_i \in C_v \cap C_{\mathcal{U}}$ implies $C_v' \cap X_i = D_i$.

Set $z_M = d_i$, if $z_{\mathcal{U}} = e_i$ for some i, and set $z_M = z_{\mathcal{U}}$ otherwise. In a similar manner, for $v \in V_{\mathcal{U}}$, set $v' = d_i$, if $v = e_{X_i}$, and set $v' = v$ otherwise. Put $V_M = \bigcup_{v \in V_{\mathcal{U}}} v'$. Now, by the infinite circuit elimination axiom (C3) applied to C_M, z_M, V_M, and $\{C_v' : v \in V_M\}$, there exists a circuit C_M' of M in $C_M \bigcup_{v' \in V_M} C_v' \setminus V_M$ such that $C_{\mathcal{U}}' = \varphi(C_M')$ is the desired set. $\square$

4.2 The (CM) axiom for localizations

The aim of this section is to prove Claim 4.8 asserting that the set system $\mathcal{C}(\mathcal{U})$ satisfies (CM), and consequently conclude our proof of Lemma 4.1 asserting that $M_{\mathcal{U}}$ is a matroid. To that end, it will be convenient for us to have a description of the independent sets and, in particular, the bases of this intended matroid. We consider this next.

Let $\mathcal{I}(\mathcal{U})$ be the set system consisting of all subsets of $E(\mathcal{U})$ not containing a member of $\mathcal{C}(\mathcal{U})$. The following describes $\mathcal{I}(\mathcal{U})$. Given an independent set $I \in \mathcal{I}(M)$ it is not hard to show that the union of the set $I \cap R(\mathcal{U})$ with the set $\{e_i : I \cap X_i \in \mathcal{B}(M|X_i)\}$ is a member of $\mathcal{I}(\mathcal{U})$. In fact, all members of $\mathcal{I}(\mathcal{U})$ are of this form.

$$\mathcal{I}(\mathcal{U}) = \{(I \cap R(\mathcal{U})) \cup \{e_i : I \cap X_i \in \mathcal{B}(M|X_i)\} : I \in \mathcal{I}(M)\}. \tag{21}$$

Proof. Given a set $I_{\mathcal{U}} \in \mathcal{I}(\mathcal{U})$, choose a base B_i of $M|X_i$ for each virtual element $e_i \in I_{\mathcal{U}}$, and set $I_M = (I_{\mathcal{U}} \cap R(\mathcal{U})) \cup \left(\bigcup_{e_i \in I_{\mathcal{U}}} B_i\right)$. We show that I_M is independent in M. Suppose not, and let C_M be a circuit of M contained in I_M. Clearly,

C_M is not contained in any member X_i of $\mathcal{U}$, for otherwise $C_M \subseteq B_i$. Hence, $C_M \in C_{\mathcal{U}}(M)$ and induces a set $C_{\mathcal{U}} \in \mathcal{C}(\mathcal{U})$ satisfying $C_{\mathcal{U}} \subset I_{\mathcal{U}}$; a contradiction.

Conversely, let I_M be an independent set in M, and consider $I_{\mathcal{U}} = (I_M \cap R(\mathcal{U})) \cup \{e_i : I_M \cap X_i \in \mathcal{B}(M|X_i)\}$. We show that $I_{\mathcal{U}} \in \mathcal{I}(\mathcal{U})$. Suppose not. Then, $I_{\mathcal{U}}$ contains a member $C_{\mathcal{U}}$ of $\mathcal{C}(\mathcal{U})$. Choose $C_M \in C_{\mathcal{U}}(M)$ such that C_M induces $C_{\mathcal{U}}$, and let $V = C_M \setminus I_M$. Observe that $C_M \cap R(\mathcal{U}) \subseteq I_M \cap R(\mathcal{U})$. Hence, if $v \in V$, then $v \in X_i$ for some i such that $I_M \cap X_i \in B(M|X_i)$. Consequently we define $C_v = C(v, I_M \cap X_i)$ for all $v \in V$. Applying the infinite circuit elimination, if necessary with two different z's, to C_M using V and $\{C_v : v \in V\}$, we obtain a circuit C'_M in I_M, a contradiction. $\square$

Next, we determine the bases of a localization. Let $\mathcal{B}(\mathcal{U})$ denote the set system consisting of the maximal members of $\mathcal{I}(\mathcal{U})$.

Lemma 4.6. $\mathcal{B}(\mathcal{U}) = \{(B \cap R(\mathcal{U})) \cup \{e_i : B \cap X_i \in \mathcal{B}(M|X_i)\} : B \in \mathcal{B}(M)\}$.

Proof. To prove the lemma, we shall use Subclaim 2 stated below. To prove the latter, we require the following.

Subclaim 1. *Let $(S, S^{\complement})$ be a 2-separation of M, and let B_S be a base of $M|S$. If $C_1, C_2 \in \mathcal{C}(M)$ satisfy $C_1 \cap S, C_2 \cap S \subseteq B_S$, then $C_1 \cap S = C_2 \cap S$.*

Proof. Suppose the claim is false and let $z \in (C_1 \cap B_S) \setminus (C_2 \cap B_S)$. By the switching lemma (see Lemma 3.8), the set $C_3 = (C_1 \cap B_2) \cup (C_2 \cap S^{\complement})$ is a circuit of M. By the circuit elimination axiom applied to C_3, z, C_2, and an element $w \in C_2 \cap S^{\complement}$, there exists a circuit C_4 contained in $C_3 \cup C_2 - w$ such that $z \in C_4$. This circuit cannot cross $(S, S^{\complement})$, for if so then $C_4 \cap S^{\complement}$ is properly contained in $C_2 \cap S^{\complement}$, a contradiction to Lemma 3.7. Then, (since $z \in C_4$) we have that $C_4 \subseteq S$ implying that $C_4 \subseteq B_S$, which is a contradiction as well. $\square$

Subclaim 2. *Let $(S, S^{\complement})$ be a 2-separation of M. Suppose that B is a base of M such that $B \cap S$ is not a base of $M|S$. Then there does not exist a circuit C of M such that $C \cap S \subseteq B$.*

Proof. Suppose there does exist such a circuit C of M. Let B_S be a base of S containing $B \cap S$. Let $f \in B_S \setminus B$. Now $C(f, B) \cap S$ and $C \cap S$ are both contained in B_S, yet $C(f, B) \cap S \neq C \cap S$, contradicting Subclaim 1. $\square$

Given Subclaim 2, we proceed to proving Lemma 4.6 as follows. Let $B_{\mathcal{U}}$ be a maximal element of $\mathcal{I}(\mathcal{U})$. By (21), there exists an independent set I_M of M such that $B_{\mathcal{U}} = (I_M \cap R(\mathcal{U})) \cup \{e_i : I_M \cap X_i \in \mathcal{B}(M|X_i)\}$. Extend I_M into a base B_M of M, and set $B'_{\mathcal{U}} = (B_M \cap R(\mathcal{U})) \cup \{e_i : B_M \cap X_i \in \mathcal{B}(M|X_i)\}$. Then, $B'_{\mathcal{U}}$ is in $\mathcal{I}(\mathcal{U})$, by (21). As $I_M \subseteq B_M$, we have that $B_{\mathcal{U}} \subseteq B'_{\mathcal{U}}$. As $B_{\mathcal{U}}$ is maximal, the equality $B_{\mathcal{U}} = B'_{\mathcal{U}}$ holds. Thus, $B_{\mathcal{U}} = (B_M \cap R(\mathcal{U})) \cup \{e_i : B_M \cap X_i \in \mathcal{B}(M|X_i)\}$ as desired.

For the converse direction, let B_M be a base of M. Let $I_{\mathcal{U}} = (B_M \cap R(\mathcal{U})) \cup \{e_i : B_M \cap X_i \in \mathcal{B}(M|X_i)\}$. We show that $I_{\mathcal{U}}$ is in $\mathcal{B}(\mathcal{U})$. Clearly, $I_{\mathcal{U}} \in \mathcal{I}(\mathcal{U})$ since B_M is independent in M. To show that $I_{\mathcal{U}}$ is maximal in $\mathcal{I}(U)$ it is

sufficient to prove that $I_{\mathcal{U}}$ "spans" $E(\mathcal{U})$; that is, for all $e \in E(\mathcal{U}) \setminus I_{\mathcal{U}}$, the set $I_{\mathcal{U}} + e$ contains a member of $\mathcal{C}(\mathcal{U})$.

Let then $e \in E(\mathcal{U}) \setminus I_{\mathcal{U}}$; suppose, first, that $e \in R(\mathcal{U})$, and let $C_M = C_M(e, B_M)$. By Subclaim 2, $C_M \cap X_i = \emptyset$, if $e_i \notin B_{\mathcal{U}}$. So C_M induces a set $C_{\mathcal{U}} \in \mathcal{C}(\mathcal{U})$ such that $C_{\mathcal{U}} \subseteq I_{\mathcal{U}} + e$. So e is spanned by $I_{\mathcal{U}}$.

Suppose, second, that e is some virtual element e_i; so that $B_M \cap X_i$ is not a base of $M|X_i$. Choose $f \in E(M)$ such that $B \cap X_i + f \in \mathcal{B}(M|X_i)$. Let $C_M = C(f, B_M)$. Now $C_M \in C_{\mathcal{U}}(M)$. Hence C_M induces a set $C_{\mathcal{U}} \in \mathcal{C}(\mathcal{U})$. By Subclaim 2, $C_M \cap X_I = \emptyset$, if $e_i \notin I_{\mathcal{U}} + e$. Hence, $C_{\mathcal{U}} \subseteq I_{\mathcal{U}} + e$. Thus, $I_{\mathcal{U}}$ spans e. □

We conclude this section by proving that $\mathcal{C}(U)$ satisfies the (CM) axiom, and consequently completing our proof of Lemma 4.1. We shall require the following.

Lemma 4.7. *Let $(S, S^{\complement})$ be a 2-separation of M, and let $X \subseteq E(M)$ such that $|S \cap X|, |S \cap X^{\complement}| \geq 2$. Then $(S \cap X, S \cap X^{\complement})$ is a 2-separation of $M|X$.*

Claim 4.8. *$\mathcal{C}(\mathcal{U})$ satisfies (CM).*

Proof. Let $A_{\mathcal{U}}$ be a subset of $E(\mathcal{U})$ and let $I_{\mathcal{U}}$ be a member of $\mathcal{I}(\mathcal{U})$ contained in $A_{\mathcal{U}}$. We are to show that $I_{\mathcal{U}}$ is contained in a maximal member of $\{I \in \mathcal{I}(\mathcal{U}) : I \subseteq A_{\mathcal{U}}\}$. To that end, let $A_M \subseteq E(M)$ be given by $A_M = \{x : \varphi_{\mathcal{U}}(x) \in A_{\mathcal{U}}\}$; this set consists of $A_{\mathcal{U}} \cap R(\mathcal{U})$ together with the members X_i for each virtual element e_i present in $A_{\mathcal{U}}$. Finally, let $I_M \in \mathcal{I}(M)$ giving rise to $I_{\mathcal{U}}$ per (21).

Extend $I_M \cap A_M$ into a base B_{A_M} of $M|A_M$, and put

$$B_{A_{\mathcal{U}}} = \{B_{A_M} \cap A_{\mathcal{U}}\} \cup \{e_i : B_{A_M} \cap X_i \in \mathcal{B}(M|X_i)\}.$$

We show that $B_{A_{\mathcal{U}}}$ is the required maximal member of $\{I \in \mathcal{I}(\mathcal{U}) : I \subseteq A_{\mathcal{U}}\}$. Clearly, $I_{\mathcal{U}} \subseteq B_{A_{\mathcal{U}}} \subseteq A_{\mathcal{U}}$. To see that $B_{A_{\mathcal{U}}} \in \mathcal{I}(\mathcal{U})$, extend B_{A_M} into a base B_M of M and note that

$$B_{A_{\mathcal{U}}} \subseteq \{B_M \cap R(\mathcal{U})\} \cup \{e_i : B_M \cap X_i \in \mathcal{B}(M|X_i)\} \underset{(21)}{\in} \mathcal{I}(\mathcal{U}).$$

To show that $B_{A_{\mathcal{U}}}$ is maximal in the required sense, it is sufficient to show that $B_{A_{\mathcal{U}}}$ "spans" $A_{\mathcal{U}}$ in the sense that $B_{A_{\mathcal{U}}} + e$ contains a member of $\mathcal{C}(\mathcal{U})$ whenever $e \in A_{\mathcal{U}} \setminus B_{A_{\mathcal{U}}}$. To see this, consider, first, an element $e \in A_{\mathcal{U}} \setminus B_{A_{\mathcal{U}}}$ that is real, i.e., $e \in R(\mathcal{U})$. In this case, the circuit $C_{M|A_M}(e, B_{A_M})$ gives rise to a member of $\mathcal{C}(\mathcal{U})$ in $B_{A_{\mathcal{U}}}$, contradicting the fact that the latter is a member of $\mathcal{I}(\mathcal{U})$.

Suppose then that e is some virtual element e_i. Consider $(X_i, A_M \setminus X_i)$. By definition, $|X_i| \geq 2$. As $I_{\mathcal{U}}$ is nonempty and does not contain e_i, by assumption, we have that $|A_M \setminus X_i| \geq 1$. We may, in fact, assume that $|A_M \setminus X_i| \geq 2$, for otherwise $1 = |I_{\mathcal{U}}| \leq |A_{\mathcal{U}}| \leq 2$ and the claim is trivially true. Consequently, $(X_i, A_M \setminus X_i)$ is a 2-separation of $M|A_M$, by Lemma 4.7. Now, if $M|A_M$ contains a circuit that crosses $(X_i, A_M \setminus X_i)$, then such a circuit gives rise to a member of $\mathcal{C}(\mathcal{U})$ that is contained in $B_{A_{\mathcal{U}}}$ which is a contradiction. Suppose then that no

circuit of $M|A_M$ crosses $(X_i, A_M \setminus X_i)$ so that $M|A_M$ is disconnected. Recall now that $e_i \notin B_{A_\mathcal{U}}$ since $B_{A_M} \cap X_i$ is not a base of $M|X_i$. These two facts imply that B_{A_M} can be extended in X_i without picking up a circuit of $M|A_M$ contradicting the assumption that it is a base of $M|A_M$. $\square$

4.3 Good 2-separations of localizations

In this section, we prove Lemma 4.10 stated below. This lemma essentially asserts that a good 2-separation of a localization $M_\mathcal{U}$ of M gives rise to a good 2-separation of M. As already mentioned, in the sequel, we shall use this lemma to prove that torsos admit no good 2-separations.

Suppose $(S, S^\complement)$ is a 2-separation of M. In Lemma 4.9 (below), we shall see that $(\varphi_\mathcal{U}(S), \varphi_\mathcal{U}(S^\complement)$ is a 2-separation of $M_\mathcal{U}$; this maps 2-separations of M to 2-separations of $M_\mathcal{U}$. In order to map 2-separations of $M_\mathcal{U}$ to 2-separations of M, we define

$$\varphi_\mathcal{U}^{-1}(Z) := \{y \in E(M) | \varphi_\mathcal{U}(y) \in Z\},$$

where $Z \subseteq E(M_\mathcal{U})$. In particular, let us remark that if $e_i \in E(M_\mathcal{U})$ is the virtual element representing the member X_i of $\mathcal{U}$ in $M_\mathcal{U}$, then $\varphi_\mathcal{U}^{-1}(e_i) = X_i$. Below we prove the following, asserting a correspondence between the 2-separations of M and those of its localization $M_\mathcal{U}$. Prior to this let us set the notation that $S^\complement$ means $E(M_\mathcal{U}) \setminus S$ whenever $S \subseteq E(M_\mathcal{U})$.

Lemma 4.9. *Let $S \subseteq E(M_\mathcal{U})$ such that $|S|, |S^\complement| \geq 2$. Then, $(S, S^\complement)$ is a 2-separation of $M_\mathcal{U}$ if and only if $(\varphi_U^{-1}(S), \varphi_U^{-1}(S^\complement))$ is a 2-separation of M.*

We shall use Lemma 4.9 in order to prove the main result of this section which reads as follows.

Lemma 4.10. *Let $(S, S^\complement)$ be a 2-separation of $M_\mathcal{U}$. Then, $(S, S^\complement)$ is a good 2-separation of M if and only if $(\varphi_\mathcal{U}^{-1}(S), \varphi_\mathcal{U}^{-1}(S^\complement))$ is a good 2-separation of M.*

We now prove Lemmas 4.9 and 4.10, starting with the former. To this end, we require some preparation. For a set $A \subseteq E(M_\mathcal{U})$, the set system

$$\mathcal{U}_A = \{X_i \in \mathcal{U} | \ \varphi_\mathcal{U}(X_i) \in A\}$$

consists of those members X_i of $\mathcal{U}$ that are mapped to some member of A by $\varphi_\mathcal{U}$; so that if A contains no virtual elements, then $\mathcal{U}_A$ is empty. Consider now the matroid $M|\varphi_\mathcal{U}^{-1}(A)$; the ground set of which is

$$\bigcup_{X \in \mathcal{U}_A} X \cup (A \cap R(\mathcal{U})).$$

A pair $(X, \varphi_\mathcal{U}^{-1}(A) \setminus X)$ where $X \in \mathcal{U}_A$ satisfying $|\varphi_\mathcal{U}^{-1}(A) \setminus X| \geq 2$, forms a 2-separation of $M|\varphi_\mathcal{U}^{-1}(A)$, by Lemma 4.7. Consequently, $(M|\varphi_\mathcal{U}^{-1}(A))_{\mathcal{U}_A}$ is a localization of $M|\varphi_\mathcal{U}^{-1}(A)$ at $\mathcal{U}_A$ provided $|\varphi_\mathcal{U}^{-1}(A) \setminus X| \geq 2$ holds for every $X \in \mathcal{U}_A$. The next claim asserts that this localization is simply the matroid $M_\mathcal{U}|A$.

Claim 4.11. *Let $A \subseteq M_{\mathcal{U}}$ such that $(M|\varphi_{\mathcal{U}}^{-1}(A))_{\mathcal{U}_A}$ is a localization (and hence a matroid). Then, $M_{\mathcal{U}}|A = (M|\varphi_{\mathcal{U}}^{-1}(A))_{\mathcal{U}_A}$.*

Proof. These two matroids have the same ground set; it suffices now to show that these have the same circuits. Let, then, C be a circuit of $M_{\mathcal{U}}|A$, and let C_M be a circuit of M satisfying $\varphi_{\mathcal{U}}(C_M) = C$. Hence, $C_M \subseteq \varphi_{\mathcal{U}}^{-1}(A)$, so that C_M is a circuit of $M|\varphi_{\mathcal{U}}^{-1}(A)$. Thus $C' = \varphi_{\mathcal{U}_A}(C_M)$ is a circuit of $(M|\varphi_{\mathcal{U}}^{-1}(A))_{\mathcal{U}_A}$. But $C' = C$.

Let, now, C be a circuit of $(M|\varphi_{\mathcal{U}}^{-1}(A))_{\mathcal{U}_A}$, and let C_M be a circuit of $M|\varphi_U^{-1}(A)$ satisfying $\varphi_{\mathcal{U}_A}(C_M) = C$. Then, C_M is a circuit of M such that $C_M \subseteq \varphi_{\mathcal{U}}^{-1}(A)$. Thus, $C' = \varphi_{\mathcal{U}}(C_M)$ is a circuit of M such that $C' \subseteq A$. But $C' = C$. $\square$

We are now ready to prove Lemma 4.9.

Proof of Lemma 4.9. Let S be a subset of $E(M_{\mathcal{U}})$ such that $|S|, |S^{\complement}| \geq 2$. Suppose, first, that $(\varphi_{\mathcal{U}}^{-1}(S), \varphi_{\mathcal{U}}^{-1}(S^{\complement}))$ is a 2-separation of M. We show that $(S, S^{\complement})$ is a 2-separation of $M_{\mathcal{U}}$. By Claim 4.11,

$$M_{\mathcal{U}}|S = (M|\varphi_{\mathcal{U}}^{-1}(S))_{\mathcal{U}_S} \text{ and } M_{\mathcal{U}}|S^{\complement} = (M|\varphi_{\mathcal{U}}^{-1}(S^{\complement}))_{\mathcal{U}_{S^{\complement}}}.$$

Let $B_{1,\mathcal{U}}$ be a base of $M_{\mathcal{U}}|S$ and $B_{2,\mathcal{U}}$ be a base of $M_{\mathcal{U}}|S^{\complement}$. As $B_{1,\mathcal{U}}$ is also a base of $(M|\varphi_{\mathcal{U}}^{-1}(S))_{\mathcal{U}_S}$, there exists a corresponding base B_1 of $M|\varphi_{\mathcal{U}}^{-1}(S)$ as in Lemma 4.6. Similarly there exists a corresponding base B_2 of $M|\varphi_{\mathcal{U}}^{-1}(S^{\complement})$ for $B_{2,\mathcal{U}}$. Let B be a base of M such that $B \subseteq B_1 \cup B_2$. As $(\varphi_{\mathcal{U}}^{-1}(S), \varphi_{\mathcal{U}}^{-1}(S^{\complement}))$ is a 2-separation of M, $|(B_1 \cup B_2) \setminus B| = 1$. Let e be this element. Consider the corresponding base $B_{\mathcal{U}}$ of $M_{\mathcal{U}}$ for B given by Lemma 4.6. Surely, $B_{\mathcal{U}} \subseteq B_{1,\mathcal{U}} \cup B_{2,\mathcal{U}}$. Indeed, deleting $\varphi_{\mathcal{U}}(e)$ from $B_{1,\mathcal{U}} \cup B_{2,\mathcal{U}}$ must give $B_{\mathcal{U}}$. Thus $(S, S^{\complement})$ is a 2-separation.

Suppose, second, that $(S, S^{\complement})$ is a 2-separation of $M_{\mathcal{U}}$. We show that $(\varphi_{\mathcal{U}}^{-1}(S), \varphi_{\mathcal{U}}^{-1}(S^{\complement}))$ is a 2-separation of M. Observe first that $|\varphi_{\mathcal{U}}^{-1}(S)|$ and $|\varphi_{\mathcal{U}}^{-1}(S^{\complement})|$ are both at least 2. Let B_1 be a base of $M|\varphi_{\mathcal{U}}^{-1}(S)$ and B_2 be a base of $M|\varphi_{\mathcal{U}}^{-1}(S^{\complement})$. Consider the corresponding bases $B_{1,\mathcal{U}}$ and $B_{2,\mathcal{U}}$ in $M_{\mathcal{U}}|S_{\mathcal{U}}$ and $M_{\mathcal{U}}|S_{\mathcal{U}}^{\complement}$. Let $B_{\mathcal{U}}$ be a base of $M_{\mathcal{U}}$ such that $B_{\mathcal{U}} \subseteq B_{1,\mathcal{U}} \cup B_{2,\mathcal{U}}$. As $S_{\mathcal{U}}$ is a 2-separation of $M_{\mathcal{U}}$, $|(B_{1,\mathcal{U}} \cup B_{2,\mathcal{U}}) \setminus B_{\mathcal{U}}| = 1$. Let e be this element of $M_{\mathcal{U}}$. Consider the corresponding base B of M for $B_{\mathcal{U}}$ such that $B \subseteq B_1 \cup B_2$. It follows that for all $f \in (B_1 \cup B_2) \setminus B$, $\varphi_{\mathcal{U}}(f) = e$. Without loss of generality, suppose that $e \in S_{\mathcal{U}}$. Let X_i be the corresponding 2-separation from $\mathcal{U}$. Thus every such f is in $\varphi_{\mathcal{U}}^{-1}(S)$. So f is in $B_1 \setminus B$. But there is only one such f because $B_1 \cap X_i$ is a base of $M|X_i$, but $B \cap X_i$ is the base of a hyperplane of $M|X_i$. Hence, $|(B_1 \cup B_2) \setminus B| = 1$ and $(S, S^{\complement})$ is a 2-separation of M. For all i, if $e_i \in S_{\mathcal{U}}$, then $X_i \subseteq S$. Similarly if $e_i \notin S_{\mathcal{U}}$, then $X_i \subseteq S^{\complement}$. Hence, $(S, S^{\complement})$ is a 2-separation of $M_{\mathcal{U}}$. $\blacksquare$

By Lemma 4.9, if $(S, S^{\complement})$ is a 2-separation of M, then $(\varphi_{\mathcal{U}}(S), \varphi_U(S^{\complement})$ is a 2-separation of $M_{\mathcal{U}}$ provided the latter two sets both have size at least 2. In fact, a stronger property holds; the next lemma asserts that a pair $(S_{\mathcal{U}}, (S_{\mathcal{U}})^{\complement})$ with $S_{\mathcal{U}} \subseteq \varphi_{\mathcal{U}}(S)$ is also a 2-separation of $M_{\mathcal{U}}$.

Lemma 4.12. *Let $(S, S^{\complement})$ be a 2-separation of a connected matroid M. If $S_{\mathcal{U}} \subseteq \varphi_{\mathcal{U}}(S)$ and $|S_{\mathcal{U}}|, |(S_{\mathcal{U}})^{\complement}| \geq 2$, then $(S_{\mathcal{U}}, (S_{\mathcal{U}})^{\complement})$ is a 2-separation of $M_{\mathcal{U}}$.*

Proof. It is sufficient to prove that

$$\text{there exists a 2-separation } (S', S'^{\complement}) \text{ of } M \text{ satisfying } \varphi_{\mathcal{U}}(S') = S_{\mathcal{U}}; \tag{22}$$

observe that for such an S' we have that $(S_{\mathcal{U}})^{\complement} \subseteq \varphi_{\mathcal{U}}(S'^{\complement})$. Assuming (22), the lemma follows via Lemma 4.9.

To prove (22), fix an ordering $(e_\alpha)_{\alpha<\gamma}$ on the elements of $M_{\mathcal{U}}$ in $\varphi_{\mathcal{U}}(S) \setminus S_{\mathcal{U}}$. We prove the following claim.

Subclaim 1. *There exists a sequence $(S_\alpha)_{\alpha<\gamma}$ of subsets of $E(M)$ such that for every α*

(i) *the pair $(S_\alpha, (S_\alpha)^{\complement})$ is a 2-separation of M;*

(ii) *$S_{\mathcal{U}} \subseteq \varphi_{\mathcal{U}}(S_\alpha)$; and*

(iii) *$S_\alpha \subseteq S_\beta$ and $e_\beta \notin \varphi_{\mathcal{U}}(S_\alpha)$ for all $\beta < \alpha$.*

Proof. We use transfinite induction. Let $S_1 = S$. As the claim holds for $\alpha = 1$, $\alpha > 1$ and that the set S_β has been defined for each $\beta < \alpha$.

If α is a successor ordinal, set

$$S_\alpha = S_{\alpha-1} \cap (\varphi_{\mathcal{U}}^{-1}(e_{\alpha-1}))^{\complement}.$$

As $e_{\alpha-1} \in \varphi_{\mathcal{U}}(S) \setminus S_{\mathcal{U}}$, the set $\varphi_{\mathcal{U}}^{-1}(e_{\alpha-1})$ intersects both S and $S^{\complement}$. Hence, $\varphi_{\mathcal{U}}^{-1}(e_{\alpha-1})$ has at least two elements and thus $e_{\alpha-1}$ must be virtual. So

$$(\varphi_{\mathcal{U}}^{-1}(e_{\alpha-1}), (\varphi_{\mathcal{U}}^{-1}(e_{\alpha-1}))^{\complement})$$

is a 2-separation of M. By the corner lemma (Lemma 3.2), $(S_\alpha, (S_\alpha)^{\complement})$ is a 2-separation of M. Note that $\varphi_{\mathcal{U}}(S_\alpha) = \varphi_{\mathcal{U}}(S_{\alpha-1}) - e_{\alpha-1}$ and that in this case the claim follows.

Suppose then that α is a limit ordinal, and set $S_\alpha = \bigcap_{\beta<\alpha} S_\beta$. By the infinite nested intersection lemma (see Lemma 3.4), $(S_\alpha, (S_\alpha)^{\complement})$ is a 2-separation of M. Moreover, $S_{\mathcal{U}} \subseteq S_\alpha$ because $S_{\mathcal{U}} \subseteq S_\beta$ for all $\beta < \alpha$ by induction. Note that $S_\alpha \subseteq S_\beta$ for all $\beta < \alpha$. For all $\beta < \alpha$, as $e_\beta \notin S_{\beta+1}$, then $e_\beta \notin S_\alpha$ and the claim follows in this case as well. □

Now $(S_\gamma, (S_\gamma)^{\complement})$ is a 2-separation of M such that $S_{\mathcal{U}} \subseteq \varphi_{\mathcal{U}}(S_\gamma)$ and $e \notin \varphi_{\mathcal{U}}(S_\gamma)$ for all $e \in \varphi_{\mathcal{U}}(S) \setminus S_{\mathcal{U}}$. Hence $S_{\mathcal{U}} = \varphi_{\mathcal{U}}(S_\gamma)$ as desired and (22) follows. □

We are now ready to prove Lemma 4.10.

Proof of Lemma 4.10. If $(S, S^{\complement})$ is not a good 2-separation of $M_{\mathcal{U}}$, then there exists a 2-separation $(S', S'^{\complement})$ of $M_{\mathcal{U}}$ crossing $(S, S^{\complement})$. By Lemma 4.9, $(\varphi_{\mathcal{U}}^{-1}(S'), \varphi_{\mathcal{U}}^{-1}(S'^{\complement}))$ is a 2-separation of M; this 2-separation crosses $(\varphi_{\mathcal{U}}^{-1}(S), \varphi_{\mathcal{U}}^{-1}(S^{\complement}))$. Thus the latter separation is not good.

If $(\varphi_{\mathcal{U}}^{-1}(S), \varphi_{\mathcal{U}}^{-1}(S^{\complement}))$ is not a good 2-separation of M, then there exists a 2-separation $(S', S'^{\complement})$ of M crossing it. Let $S'_{\mathcal{U}}$ be a subset of $\varphi_{\mathcal{U}}(S')$ such that $(S'_{\mathcal{U}})^{\complement}$ is a subset of $\varphi_{\mathcal{U}}(S'^{\complement})$ and $(S'_{\mathcal{U}}, (S'_{\mathcal{U}})^{\complement})$ crosses $(S, S^{\complement})$. By Lemma 4.12, $(S'_{\mathcal{U}}, (S'_{\mathcal{U}})^{\complement})$ is a proper 2-separation of M. Hence, $(S, S^{\complement})$ is not good. ■

5 Nested sequences of good 2-separations

In this section we prove Proposition 2.1 asserting that *the intersection of an infinite sequence of nested good 2-separations is always empty*. We do not know whether the lemma extends to k-separations for $k > 2$.

Proof of Proposition 2.1. Assume towards contradiction that $S_{\cap} = \bigcap_{i=1}^{\infty} S_i \neq \emptyset$. We may assume that $|S_{\cap}| = \{e\}$. Indeed, if $|S_{\cap}| \geq 2$, then $(S_{\cap}, (S_{\cap})^{\complement})$ is a 2-separation by the infinite nested intersection lemma (see Lemma 3.4); so that $M = M_1 \oplus_2 M_2$, by Lemma 3.6, where M_1 has $E(M) \setminus (\bigcap_{i=1}^{\infty} S_i)$ plus an additional element e introduced by the 2-sum operation (see Lemma 3.6) as its ground set. M_1 then contains an infinite nested sequence $\{S'_1 \supsetneq S'_2 \supsetneq S'_3 \ldots\}$ where $S'_i = (S_i + e) \setminus (E(M_2) - e)$ and $(S'_i, (S'_i)^{\complement})$ is a good 2-separation of M_1 for every i'. Clearly, $\bigcap_{i'=1}^{\infty} S_{i'} = \{e\}$. Hence, we may put $M = M_1$ and proceed assuming that $S_{\cap} = \{e\}$.

In what follows, we show that M is not a matroid. We establish this by constructing a base B of $M - e$ such that $B + e$ is independent M. This then shows that M is not connected; a contradiction. Choose a circuit C containing e. As M is connected, such a circuit exists and satisfies $C \neq \{e\}$. In fact,

$$C - e \subseteq S_i^{\complement} \text{ for no } i. \tag{23}$$

Proof. Assume towards contradiction that $C - e \subseteq S_i^{\complement}$ for some i. Let B_1 be a base of $M|S_i^{\complement}$ containing $C - e$; such is a base of $M|S_i^{\complement}$ (as clearly B_1 spans e). Next, let B_2 be a base of $M|(S_i - e)$ and let B'_2 be a base of $M|S_i$ containing B_2; clearly $B'_2 \subseteq B_2 + e$.

As $(S_i, S_i^{\complement})$ is a 2-separation of M, there exists an element $f \in B_1 \cup B'_2$ such that $B_1 \cup B'_2 - f$ is a base of M. If $B'_2 = B_2 + e$, then $f \in C$ and we may assume that $f = e$. In this case, $(S_i^{\complement} + e, S_i - e)$ is a 1-separation of M a contradiction to M being connected. Suppose then that $B'_2 = B_2$. Then, $(S_i^{\complement} + e, S_i - e)$ is a 2-separation of M crossing $(S_{i+1}, (S_{i+1})^{\complement})$; a contradiction. □

Let $L_1 = S_1^{\complement}$ and put $L_i = S_i^{\complement} - S_{i-1}^{\complement}$ for $i \geq 2$. We refer to L_i as the ith *block* of M. A corollary of (23) is that there exist infinitely many i for which $C \cap L_i \neq \emptyset$. Without loss of generality (by possibly discarding some of the S_i's

and redefining the blocks), we may assume that

$$C \cap L_i \neq \emptyset \text{ for all } i. \tag{24}$$

Let $U_1 = \{S_1\}$ and put $U_i = \{S_i, (S_{i-1})^{\complement}\}$ for $l \geq 2$. Let M_i denote the localization M_{U_i}; such satisfies $R(M_i) = L_i$ for every i and is called *real* if $R(M_i)$ spans M_i. We write e_1 to denote the virtual element of M_1 and $\{e_{i-1}, e_i\}$ for those of M_i where for $i \geq 2$. Next, for $i \geq 1$ call $M_i \oplus_2 M_{i+1}$ *real* whenever such is spanned by $E(M_i \oplus_2 M_{i+1}) \setminus \{e_{i-1}, e_{i+1}\}$; naturally if $i = 1$ we take $E(M_1 \oplus_2 M_2) - e_2$ instead. We show that

$$M_i \oplus_2 M_{i+1} \text{ is real } \iff M_i \text{ or } M_{i+1} \text{ is real.} \tag{25}$$

Proof. Suppose M_i is real. Let B_i be a base of M_i contained in L_i, and let B_{i+1} be a base of $M_{i+1} \setminus e_{i+1}$ containing e_i. As $e_i, e_{i+1} \in \varphi_{U_{i+1}}(C)$, there is a circuit in M_{i+1} containing e_i and e_{i+1}. Consequently, B_{i+1} is a base of M_{i+1}. Put $B = B_i \cup (B_{i+1} - e_i)$ and note that this is a base of $M_i \oplus_2 M_{i+1}$ such that $e_{i-1}, e_{i+1} \notin B$. Hence, $M_i \oplus_2 M_{i+1}$ is real as desired. An analogous argument holds when M_{i+1} is local.

Suppose then that $M_i \oplus_2 M_{i+1}$ is real. Let B be a base of $M_i \oplus_2 M_{i+1}$ containing neither of e_{i-1} and e_{i+1}. As $(S_i, S_i^{\complement})$ is a 2-separation of M, $(E(M_i) - e_i, E(M_{i+1}) - e_i)$ is a 2-separation of $M_i \oplus_2 M_{i+1}$, by Lemma 4.9. Thus, either $B_1 = B \cap E(M_i) - e_i$ is a base of $M_i - e_i$ or $B_2 = B \cap E(M_{i+1}) - e_i$ is a base of $M_{i+1} - e_i$. In the former case, B_1 is also a base of M_i and yet $e_{i-1}, e_i \notin B_1$ as desired. Similarly in the latter case, B_2 is a base of M_{i+1} and yet $e_i, e_{i+1} \notin B_2$ as desired. □

We may now prove that

$$M_i \text{ or } M_{i+1} \text{ is real, for any } i. \tag{26}$$

Proof. Suppose towards contradiction that M_i and M_{i+1} are not real. Then, $M' = M_i \oplus_2 M_{i+1}$ is not real, by (25). As every base of M' includes one of e_{i-1}, e_{i+1} and as as L_i and L_{i+1} are nonempty, by (24), it follows that $(\{e_{i-1}, e_{i+1}\}, L_i \cup L_{i+1})$ is a 2-separation of M.

By Lemma 4.9, $(E(M) - (L_i \cup L_{i+1}), L_i \cup L_{i+1})$ is a 2-separation of M. As such crosses $(S_i, S_i^{\complement})$ we attain a contradiction to the assumption that $(S_i, S_i^{\complement})$ is good. □

A corollary of (26) is that there exist infinitely many i such that M_i is real. Without loss of generality (by discarding some of the S_i's and redefining blocks), we may assume, by (25), that

$$M_i \text{ is real, for all } i. \tag{27}$$

Let $B_1 \in \mathcal{B}(M_1)$ such that $e_1 \in B_1$. For all $k \geq 1$, let $B_{2k} \in \mathcal{B}(M_{2k}$ such that $e_{2k-1}, e_{2k} \notin B_{2k}$; such exists as M_{2k} is real. For all $k \geq 1$, let $B_{2k+1} \in \mathcal{B}(M_{2k+1})$ containing $\{e_{2k}, e_{2k+1}\}$. Such a base exist as $\{e_{2k}, e_{2k+1}\}$ is independent in M_{2k}.

The latter follows from the fact that $\{e_{2k}, e_{2k+1}\}$ is a proper subset of $\varphi_{U_{2k+1}}(C)$ as $C \cap L_{2k+1} \neq \emptyset$. Define $B = \bigcup_i B_i \cap L_i$, and observe that

$$B + e \text{ is independent.} \tag{28}$$

Proof. Suppose not. Then there exists a circuit C of M contained in $B + e$. Let i be minimum such that $C \cap L_i \neq \emptyset$. Note that either two such i exists as $B \cap L_i = B_i \cap L_i$ is independent, or $e \in C$. If i is odd, then $C' = \varphi_{U_i}(C)$ has at least two elements, one in L_i and e_i. Hence, C' is a circuit of M_i contained in $L_i + e_i \subseteq B_i$, a contradiction. If i is even, then $C' = \varphi_{U_{i+1}}(C)$ has at least two elements, e_i and either one in L_i or e_{i+1}. Hence, C' is a circuit of M_{i+1} contained in $L_i + e_i + e_{i+1} \subseteq B_{i+1}$, a contradiction. $\square$

To conclude we show that

$$B \text{ spans } M - e. \tag{29}$$

Proof. Let $v \in M - e$ such satisfies $v \in L_i$ for some i. To show that B spans v, we prove that $B + v$ has a circuit C containing v. Consider the fundamental circuit $C(v, B_i)$ in M_i. If $C(v, B_i) \subseteq L_i$, then $C(v, B_i) \subseteq B_i \cap L_i + v \subseteq B + v$ as desired. Assume then that $C(v, B_i) \setminus L_i \neq \emptyset$. Thus, i is odd as otherwise $B_i \subseteq L_i$. If $i = 1$, then $C(v, B_1) \setminus L_1 \subseteq \{e_1\}$ and hence $e_1 \in C(v, B_1)$. But then $C(v, B_1) \cup C(e_1, B_2) - e_1$ is a circuit of M contained in $B_1 \cup B_2 + v - e_1 \subseteq B + v$ as desired.

Suppose then that $i \neq 1$. Then, $C(v, B_i) \setminus L_i \subseteq \{e_{i-1}, e_i\}$. If $C(v, B_i) \setminus L_i = e_{i-1}$, then $C(e_{i-1}, B_{i-1}) \cup C(v, B_i) - e_{i-1}$ is a circuit of M as desired. Similarly if $C(v, B_i) \setminus L_i = e_i$, , then $C(v, B_i) \cup C(e_i, B_{i+1}) - e_i$ is a circuit of M as desired. Finally if if $C(v, B_i) \setminus L_i = \{e_{i-1}, e_i\}$, then $C(e_{i-1}, B_{i-1}) \cup C(v, B_i) \cup C(e_i, B_{i+1}) - e_{i-1} - e_i$ is a circuit of M as desired. $\square$

By (28), B is a base of $M - e$. However, by (29), $B + e$ is independent. Thus, M is not connected, a contradiction. This proves Proposition 2.1. $\blacksquare$

6 The structure of torsos

In this section we prove Lemma 2.3; recall that this lemma asserts that *a connected matroid with no good 2-separations is* 3*-connected, a circuit, or a circuit*, and note that the converse of this lemma is trivial. As mentioned in Section 2, our proof of Lemma 2.3 is carried out in two steps; these are captured by Lemmas 2.5 and 2.4 that together imply Lemma 2.3. This general two step framework is that of Cunningham and Edmonds [6]. The proof of Lemma 2.5 is simple. The proof of Lemma 2.4, however, requires effort and new ideas in order to be hold for infinite matroids.

Lemma 2.5 is a consequence of [8, Corollary 8.1.11]. As in [8] a proof of the latter is not provided, we include one here for completeness.

Lemma 6.1. [8, Corollary 8.1.11]
If $(S, S^{\complement})$ is a k-separation of a k-connected matroid M with $|S| = k$, then S is either a coindenpendent circuit or an independent cocircuit.

Proof. Suppose that S is coindependent. Then, $S^{\complement}$ spans M and contains a base B of M. Let B_S be a base of $M|S$. As M is k-connected and $(S, S^{\complement})$ is a k-separation, we must remove exactly $k-1$ elements from $B \cup B_S$ in order to obtain a base of M. Thus $|B_S| = k-1$. As $|S| = k$, S must be a circuit of size k. Similarly if S is independent then S is a cocircuit.

So we may assume that S is dependent and codependent. Note that a k-connected matroid with $|E(M)| \geq 2(k-1)$ has all its circuits and cocircuits of size $\geq k$ as a circuit of size j is a j-separation. Since a k-connected matroid admitting a proper k-separation satisfies $|E(M)| \geq 2k$, it follows that S is a circuit and a cocircuit. Let $v \in S$. As $S - v$ is conindependent, $S^{\complement} + v$ contains a base B of M. As S is not coindependent, $v \in B$. Let $u \neq v$ with $u \in S$. Note that $S - u$ is independent but S is not. Hence, $S - u$ is a base of S. Meanwhile, $B - v$ is a base of $S^{\complement}$. However, $|(B - v) \cup (S - u) \setminus B| = k-2$. So $(S, S^{\complement})$ is a $(k-1)$-separation, contradicting that M is k-connected. □

We are now ready to prove Lemma 2.5.

Proof of Lemma 2.5. By assumption, $(\{x, y\}, \{x, y\}^{\complement})$ is a 2-separation of M for each pair $\{x, y\} \subseteq E(M)$. By Lemma 2.5, every such pair is then either a circuit or a cocircuit. As circuit C and a cocircuit C^* never satisfy $|C \cap C^*| = 1$ [4, Lemma 3.1], either every pair is a circuit or every pair is a cocircuit; hence M is either a cocircuit or a circuit, respectively. ■

6.1 Non-3-connected primitive matroids

In this section, we prove Lemma 2.4. Let us call a connected matroid *primitive* if it has no good 2-separations. With this terminology, Lemma 2.4 reads as follows.

Lemma 6.2. *If M is a primitive matroid that is not 3-connected, then for every two elements x, y the partition $(\{x, y\}, \{x, y\}^{\complement})$ is a 2-separation.*

For the remainder of this section, *M denotes a primitive matroid that is not 3-connected.* The goal of this section then is to show that $(\{x, y\}, \{x, y\}^{\complement})$ is a 2-separation of M for every pair of its elements x and y. The first step in our proof is Lemma 6.4 stated below; this lemma asserts that for any pair of elements x and y, the matroid M admits a 2-separation with x and y on opposite sides of the separation. We shall see that Lemma 6.4 is a consequence of the following lemma.

Lemma 6.3. *Let $X \subseteq E(M)$, $|X| \geq 2$ and (S, S^C) be a 2-separation such that $X \subseteq S$. Then, there exists two crossing 2-separations $(S', S'^{\complement})$ and $(U, U^{\complement})$ satisfying $X \subseteq S' \subseteq S$ and $X \cap U, X \cap U^{\complement} \neq \emptyset$.*

Proof. Suppose to a contradiction that there is no such pair of 2-separations. Fix an ordinal $\alpha > 1$ and suppose $S_{\alpha'}$ has been defined for every ordinal $\alpha' < \alpha$. Put

$$S_\alpha = \begin{cases} S_{\alpha-1} \cap U & \text{whenever } \alpha \text{ is a successor ordinal;} \\ \bigcap_{\alpha'<\alpha} S_{\alpha'} & \text{whenever } \alpha \text{ is a limit ordinal,} \end{cases} \tag{30}$$

where $(U, U^{\complement})$ is a 2-separation crossing $(S_{\alpha-1}, (S_{\alpha-1})^{\complement})$ and $X \subseteq U$.

The pair $(U, U^{\complement})$ exists as long as $(S_{\alpha-1}, (S_{\alpha-1})^{\complement})$ is a 2-separation; indeed, if so, then the assumption that M has no good 2-separations implies that there is a 2-separation, namely $(U, U^{\complement})$, crossing $(S_{\alpha-1}, (S_{\alpha-1})^{\complement})$. To this end, we prove that

$$\text{for all } \alpha,\ (S_\alpha, (S_\alpha)^{\complement}) \text{ is a 2-separation of } M. \tag{31}$$

Proof. We prove (31) by transfinite induction. If α is a successor ordinal, then $S_\alpha = S_{\alpha-1} \cap U$. By induction $(S_{\alpha-1}, (S_{\alpha-1})^{\complement})$ is a 2-separation. By definition, so is $(U, U^{\complement})$. Thus, by the corner lemma (Lemma 3.2), $(S_\alpha, (S_\alpha)^{\complement})$ is a 2-separation of M. If α is a limit ordinal, then $S_\alpha = \bigcap_{\alpha'<\alpha} S_{\alpha'}$. By induction, $(S_{\alpha'}, (S_{\alpha'})^{\complement})$ is a 2-separation for all α. By the infinite nested intersection lemma (see Lemma 3.4), $(S_\alpha, (S_\alpha)^{\complement})$ is a 2-separation of M as $|S_\alpha|, |(S_\alpha)^{\complement}| \geq 2$, and (31) follows. $\square$

Next, we prove that

$$\text{for all } \alpha > \alpha',\ S_\alpha \subsetneq S_{\alpha'}. \tag{32}$$

Proof. We prove (32) by transfinite induction. Suppose that α is a sucessor ordinal. Then as $(U, U^{\complement})$ and $(S_{\alpha-1}, S^{\complement}_{\alpha-1})$ are crossing, $S_{\alpha-1} \setminus U$ is nonempty. Thus $S_\alpha \subsetneq S_{\alpha-1}$. By induction, $S_{\alpha-1} \subsetneq S_{\alpha'}$ for all $\alpha' < \alpha - 1$ and (32) follows. Suppose that α is a limit ordinal. Consider an ordinal $\alpha' < \alpha$. By definition, $S_\alpha \subseteq S_{\alpha'+1}$. By induction, $S_{\alpha'+1} \subsetneq S_{\alpha'}$ and (32) follows. $\square$

Observe now that if for some pair $(U, U^{\complement})$ both U and $U^{\complement}$ meet X, then $(S_{\alpha-1}, (S_{\alpha-1})^{\complement})$ and $(U, U^{\complement})$ are the desired pair of separations. Consequently, we may assume that X is a subset of U or $U^{\complement}$. Without loss of generality, we assume that $X \subseteq U$. Then, It follows by transfinite induction that for all α, the t $X \subseteq S_\alpha$.

We can now derive a contradiction as we have defined a nested sequence of subsets S_α of $E(M)$ for all ordinals α, and Lemma 6.3 follows. $\square$

Lemma 6.4 is a consequence of Lemma 6.3.

Lemma 6.4. *For all distinct $u, v \in E(M)$ there is a 2-separation $(S, S^{\complement})$ satisfying $u \in S$ and $v \in S^{\complement}$.*

Proof. Let $X = \{u, v\}$. As M is not 3-connected, there exists a 2-separation $(S, S^{\complement})$ of M. If X intersects both S and $S^{\complement}$, Lemma 6.4 follows. So we may assume without loss of generality that $X \subseteq S$. Applying Lemma 6.3, there exists a 2-separation $(U, U^{\complement})$ such that X intersects both U and $U^{\complement}$ and Lemma 6.4 follows. $\square$

We shall require the following stronger property than Lemma 6.4.

Lemma 6.5. *For all distinct $x, y, z \in E(M)$ there is a 2-separation $(S, S^{\complement})$ of M satisfying $\{x, y\} \subseteq S$ and $z \in S^{\complement}$.*

Proof. By Lemma 6.4, there exists a 2-separation $(S, S^{\complement})$ satisfying $x \in S^{\complement}$ and $z \in S$. We may assume that $y \in S$ as otherwise Lemma 6.5 follows. Let $X = \{y, z\}$. Applying Lemma 6.3 to X and $(S, S^{\complement})$, there exists two crossing 2-separations $(S', S'^{\complement})$ and $(U, U^{\complement})$ such that $X \subseteq S' \subseteq S$ and $U \cap X, U \cap X^{\complement} \neq \emptyset$. We may assume without loss of generality that $y \in U$ and $z \in U^{\complement}$. Note that $x \in S'^{\complement}$. If x is in U, then $(U, U^{\complement})$ is the desired separation. So suppose that $x \in U^{\complement}$. By the symmetric difference lemma (Lemma 3.3), $(S\Delta U^{\complement}, (S\Delta U^{\complement})^{\complement})$ is a 2-separation of M with $\{x, y\} \subseteq S\Delta U^{\complement}$ and $z \in (S\Delta U^{\complement})^{\complement}$ as desired. $\square$

We now prove Lemma 2.4.

Proof of Lemma 2.4. Let M be a primitive matroid and assume towards contradiction that there exists a pair of elements, say x and y, such that $(\{x, y\}, \{x, y\}^{\complement})$ is not a 2-separation of M. Choose $u \in \{x, y\}^{\complement}$. By Lemma 6.5, there exists a 2-separation $(S_1, S_1^{\complement})$ of M such that $x, y \in S_1$ and $u \notin S_1$.

Subclaim 1. *There exists a sequence of sets $\{S_\alpha\}$ such that*

(i) *$x, y \in S_\alpha$ for every α;*

(ii) *$S_\beta \supsetneq S_\alpha$ for every $\beta < \alpha$; and*

(iii) *$(S_\alpha, (S_\alpha)^{\complement})$ is a 2-separation of M for every α.*

Proof. We prove the claim via transfinite induction. The claim holds for $\alpha = 1$; suppose then that it holds for all $\beta < \alpha$. Now, if α is a successor ordinal, then set $\mathcal{U}_\alpha = \{(S_{\alpha-1})^{\complement}\}$ and consider the localization of $M_{\mathcal{U}_\alpha}$ of M at $\mathcal{U}_\alpha$. If $M_{\mathcal{U}_\alpha}$ has a good 2-separation then so does M, by Lemma 4.10, which is a contradiction. Now, by definition of M, there exists a 2-separation $(S', S'^{\complement})$ of M crossing $(S_{\alpha-1}, (S_{\alpha-1})^{\complement})$. If $x, y \in S'$, then set $S_\alpha = S' \cap S_{\alpha-1}$. By the corner lemma (Lemma 3.2), $(S_\alpha, S_\alpha^{\complement})$ is a 2-separation of M. Moreover, $x, y \in S_\alpha$ and $S_{\alpha-1} \supsetneq S_\alpha$ and the statement follows. So we may assume by symmetry and without loss of generality that $x \in S'$ and $y \in S'^{\complement}$.

As $(\{x, y\}, \{x, y\}^{\complement})$ is not a 2-separation of M, by assumption, there exists a $v \in S_{\alpha-1} - \{x, y\}$. We may assume, without loss of generality, that $v \in S'$. Let $w = \varphi_{\mathcal{U}}(S_{\alpha-1}^{\complement})$. By Lemma 4.12, there exists a 2-separation $(S_{\mathcal{U}_\alpha}, (S_{\mathcal{U}_\alpha})^{\complement})$ of $M_{\mathcal{U}_\alpha}$ such that $x, v \in S_{\mathcal{U}_\alpha}$ and $y, w \in (S_{\mathcal{U}_\alpha})^{\complement}$. Hence, $M_{\mathcal{U}_\alpha}$ is not 3-connected. By Lemma 6.5 applied to $M_{\mathcal{U}_\alpha}$, there exists a 2-separation $((S_\alpha)_{\mathcal{U}}, (S_\alpha)_{\mathcal{U}}^{\complement})$ such that $x, y \in (S_\alpha)_{\mathcal{U}}$ and $w \in (S_\alpha)_{\mathcal{U}}^{\complement}$. Let $S_\alpha = \varphi_{\mathcal{U}_\alpha}^{-1}((S_\alpha)_{\mathcal{U}})$. By Lemma 4.9, $(S_\alpha, (S_\alpha)^{\complement})$ is a 2-separation of M. Clearly, $x, y \in S_\alpha$. Moreover, $S_{\alpha-1} \supsetneq S_\alpha$ because $|(S_\alpha)_{\mathcal{U}}^{\complement}| \geq 2$.

Suppose then that α is a limit ordinal and define $S_\alpha = \bigcap_{\beta<\alpha} S_\beta$. By Lemma 3.4, S_α is a 2-separation of M. By induction, $x, y \in S_\beta$ for all $\beta < \alpha$. Hence, $x, y \in S_\alpha$. Furthermore, by induction $S_{\beta+1} \supsetneq S_\beta$ for all $\beta < \alpha$. By definition, $S_\alpha \subseteq S_{\beta+1}$. Hence, $S_\beta \supsetneq S_\alpha$ for all $\beta < \alpha$ and the statement follows. $\square$

By Subclaim 1 there is a strictly increasing sequence of subsets of $E(M)$ for all ordinals, which is a contradiction. Lemma 2.4 follows. ■

7 Constructing a decomposition tree

The goal of this section is to prove Lemma 7.1 stated below. Prior to stating this lemma, let us first be reminded of some of the notation and terminology set in Section 2. Let $\mathcal{G} = \mathcal{G}(M)$ be a set comprised of all the good 2-separation of a connected matroid M; this is a set of nested 2-separations of M with the property that $(A, A^{\complement}) \in \mathcal{G}$ implies $(A^{\complement}, A) \in \mathcal{G}$. Define a partial ordering on $\mathcal{G}$ given by writing $(A, A^{\complement}) \leq (B, B^{\complement})$ whenever $A \subseteq B$. Next, call $(A, A^{\complement})$ and $(B, B^{\complement})$ *equivalent*, and write $(A, A^{\complement}) \sim (B, B^{\complement})$, if either $(A, A^{\complement}) = (B, B^{\complement})$ or $(A^{\complement}, A)$ is a predecessor of $(B, B^{\complement})$ in this ordering. Finally, let $T_{\mathcal{G}}$ and $R_{\mathcal{G}}$ be as defined in (3), (4), and (5).

This section is dedicated to the proof of the following lemma.

Lemma 7.1. *$(T_{\mathcal{G}}, R_{\mathcal{G}})$ is an irredundant tree-decomposition of uniform adhesion 2 with all of its torsos primitive.*

In what follows, we verify each of the properties claimed for $(T_{\mathcal{G}}, R_{\mathcal{G}})$ in Lemma 7.1. We begin by establishing that the vertices of $T_{\mathcal{G}}$ are properly defined. That is, we prove that

$$\text{the relation } \sim \text{ is an equivalence relation on } \mathcal{G}. \tag{33}$$

Proof. By definition, the relation $\sim$ is reflexive and symmetric. We prove that $\sim$ is transitive. Suppose then that $(A, A^{\complement}) \sim (B, B^{\complement})$ and that $(B, B^{\complement}) \sim (C, C^{\complement}) \in \mathcal{G}$. Assume now towards contradiction that $(A, A^{\complement}) \not\sim (C, C^{\complement})$. Then, $A \neq C$ and there exists $(D, D^{\complement}) \in \mathcal{G}$ such that $(A^{\complement}, A) > (D, D^{\complement}) > (C, C^{\complement})$. That is, $A \supsetneq D \supsetneq C^{\complement}$. Clearly, $B \neq A$ and $B \neq C$. By definition of the relation $\sim$, we have that $A \supsetneq B^{\complement}$ and $C \supsetneq B^{\complement}$. Now, D does not contain B as $A^{\complement} \subsetneq B$. Similarly $D^{\complement}$ does not contain B as $C^{\complement} \subsetneq B$. As $\mathcal{G}$ is nested, either D or $D^{\complement}$ must contain $B^{\complement}$. Without loss of generality, suppose that D contains $B^{\complement}$. As $C^{\complement}$ is a subset of D and C contains $B^{\complement}$, it follows that $D \supsetneq B^{\complement}$. But then $(A, A^{\complement}) > (D, D^{\complement}) > (B^{\complement}, B)$, a contradiction. □

Next, we prove that $T_{\mathcal{G}}$ is a tree.

Claim 7.2. *$T_{\mathcal{G}}$ is a tree.*

Proof. To prove that $T_{\mathcal{G}}$ is a tree, we show that $T_{\mathcal{G}}$ is connected and acyclic. Suppose that $T_{\mathcal{G}}$ had a cycle $v_1 v_2 \dots v_n$. Let $\{A_i, A_i^{\complement}\}$ represent the edge between v_i and v_{i+1} where values are taken modulo n. We may then assume without loss of generality that $A_i \subseteq A_{i+1}$ where values are taken modulo n. But then certainly, all these sets are equal, in which case all of the 2-separations $(A_i, A_i^{\complement})$ are incident, but this is impossible if $n \geq 3$, a contradiction. Thus $T_{\mathcal{G}}$ is acyclic.

To prove $T_{\mathcal{G}}$ is connected, we show that the unique path between any two distinct nodes $u, v \in V(T_{\mathcal{G}})$ is finite. Let $(A, A^{\complement})$ be a member of the incidence class corresponding to v such that $R_v \subseteq A$ and $R_u \subseteq A^{\complement}$. Similarly let $(B, B^{\complement})$ be a member of the incidence class corresponding to v such that $R_v \subseteq B$ and $R_u \subseteq B^{\complement}$. Consider a maximal sequence $(A, A^{\complement}) > (S_1, S_1^{\complement}) > (S_2, S_2^{\complement}) > \ldots > (B, B^{\complement})$ where, for all i, $(S_i, S_i^{\complement}) \in \mathcal{G}$ is a good 2-separation of M. By Proposition 2.1, this sequence is finite. This sequence corresponds to a path in $T_{\mathcal{G}}$. $\square$

Next, we consider the adhesion of $(T_{\mathcal{G}}, R_{\mathcal{G}})$ and prove the following.

Claim 7.3. *$(T_{\mathcal{G}}, R_{\mathcal{G}})$ has uniform adhesion 2.*

Proof. The following two claims facilitate our proof.

Subclaim 1. *The sets $(R_v)_{v \in V(T_{\mathcal{G}})}$ partition $E(M)$.*

Proof. Suppose we are given $x \in E(M)$. We want to find $v \in \mathcal{V}(T_{\mathcal{G}})$ with $x \in R_v$. To do this, orient every edge $\{A, A^{\complement}\}$ of $T_{\mathcal{G}}$ towards $[(A, A^{\complement})]$ if $x \in A$ and towards $[(A^{\complement}, A)]$ if $x \in A^{\complement}$. Let v be a sink under this orientation. A sink exists as otherwise there would be an infinite directed path $[(S_1, S_1^{\complement})], [(S_2, S_2^{\complement})], \ldots$ with $S_1 \supsetneq S_2 \subsetneq \ldots$ where for all i, $(S_i, S_i^{\complement}) \in \mathcal{G}$ is a good 2-separation and $x \in S_i$. Thus, $\bigcap_i S_i \supseteq \{e\}$, contradicting Proposition 2.1. So v exists, which implies that for all $(A, A^{\complement}) \in \mathcal{G}$ in the incidence class of v, $x \in A$. By definition then, $x \in R_v$.

Conversely, we show that $R_v \cap R_w = \emptyset$ for distinct nodes v, w. As $T_{\mathcal{G}}$ is connected by Claim 7.2, let $\{A, A^{\complement}\}$ be an edge along the path between v and w such that $R_v \subseteq A$ and $R_w \subseteq A^{\complement}$. So $R_v \cap R_w = \emptyset$. $\square$

Recall now that given an edge $e = vw$ of $T_{\mathcal{G}}$, we write T_v and T_w for the components of $T_{\mathcal{G}} - e$ containing v and w, respectively, and that $S(e, v) = \bigcup_{u \in T_v} R_u$ and that $S(e, w) = \bigcup_{u \in T_w} R_u$, where R_u and R_v are as in (5).

Subclaim 2. *For every edge $e = \{A, A^{\complement}\}$ of $T_{\mathcal{G}}$, $S(e, [(A, A^{\complement})]) = A$ and $S(e, [(A^{\complement}, A)]) = A^{\complement}$.*

Proof. First, we show that $S(e, [(A, A^{\complement})]) \subseteq A$. Let T_A denote the component of $T_{\mathcal{G}} - e$ containing the node $[(A, A^{\complement})]$. It suffices to prove that $R_u \subseteq A$ for all $u \in T_A$. We prove this by induction on the length of the path P from u to $[(A, A^{\complement})]$ in T_A. If $u = [(A, A^{\complement})]$, then by definition $R_u \subseteq A$. So we may assume that P has at least one edge. Let $f = \{B, B^{\complement}\}$ be the edge in P incident with $[(A, A^{\complement})]$. We may assume without loss of generality that $(B^{\complement}, B)$ is incident with $(A, A^{\complement})$. That is, $B \subsetneq A$. Now $P - f$ is a path from from u to $[(B, B^{\complement})]$ with smaller length than P. By induction, $R_u \subseteq B$. Hence $R_u \subseteq A$ as desired.

By symmetry $S(e, [(A^{\complement}, A)]) \subseteq A^{\complement}$. By Subclaim 1, $S(e, [(A, A^{\complement})])$ and $S(e, [(A^{\complement}, A)])$ partition $E(M)$. Hence, it follows that $S(e, [(A, A^{\complement})]) = A$ and $S(e, [(A^{\complement}, A)]) = A^{\complement}$. $\square$

Claim 7.3 now follows. $\square$

Recall that for a vertex v of $T_{\mathcal{G}}$, we write M_v to denote the torso of $(T_{\mathcal{G}}, R_{\mathcal{G}})$ associated with v. We prove the following.

Claim 7.4. *For every vertex v of $T_{\mathcal{G}}$, the torso M_v has no good 2-separations.*

Proof. Suppose that M_v has a good 2-separation $(S, S^{\complement})$, and set

$$\mathcal{U} = \{S(e, v) |\ e \text{ is an edge incident with } v\}.$$

Then, M_v is equal to the localization of M at $\mathcal{U}$. By Lemma 4.10, $(\varphi_{\mathcal{U}}^{-1}(S), \varphi_{\mathcal{U}}^{-1}(S^{\complement}))$ is a good 2-separation of M. As $|S|, |S^{\complement}| \geq 2$, this separation does not correspond to an edge of $T_{\mathcal{G}}$, a contradiction. $\square$

The irredundancy of $(T_{\mathcal{G}}, R_{\mathcal{G}})$ is considered next.

Claim 7.5. *$(T_{\mathcal{G}}, R_{\mathcal{G}})$ is irredundant.*

Proof. Suppose not. Then there exists an edge uv of $T_{\mathcal{G}}$ such that, without loss of generality, the torsos M_u and M_v are circuits. By the construction of $T_{\mathcal{G}}$, $S(e, u)$ is a good 2-separation in M. Put $\mathcal{U} = \{S(f, u) : f \neq e\} \cup \{S(f, v) : f \neq e\}$, and consider the localization $M_{\mathcal{U}}$ of M at $\mathcal{U}$. Set $S_{\mathcal{U}} = \varphi_{\mathcal{U}}(S(e, u))$. By Lemma 4.10, $(S_{\mathcal{U}}, S_{\mathcal{U}}^{\complement})$ is a good 2-separation of $M_{\mathcal{U}}$. However, $M_{\mathcal{U}}$ is simply the 2-sum of the torsos M_u and M_v along the element e. Note that the 2-sum of two circuits is a circuit and that the 2-sum of two cocircuits is a cocircuit, implying that $M_{\mathcal{U}}$ has no good 2-separations, which is a contradiction. $\square$

Finally, we show that any other irredundant tree-decomposition of M whose torsos are primitive is isomorphic to $(T_{\mathcal{G}}, R_{\mathcal{G}})$ as specified in Theorem 1.1(ii).

Claim 7.6. *$(T_{\mathcal{G}}, R_{\mathcal{G}})$ is the unique irredundant tree-decomposition of M whose torsos are primitive.*

Proof. To show uniqueness of $(T_{\mathcal{G}}, R_{\mathcal{G}})$, it suffices to prove that any irredundant decomposition tree T' of M is isomorphic to $T_{\mathcal{G}}$. We show that every edge of T' is a good 2-separation in M and that every good 2-separation in M is an edge of T'.

Suppose that there exists an edge $e = uv$ of T' such that $(S(e, u), S(e, v))$ is not a good 2-separation of M. Let $\mathcal{U} = \{S(f, u) : f \neq e\} \cup \{S(f, v) : f \neq e\}$, and consider the localization $M_{\mathcal{U}}$ of M at $\mathcal{U}$. By Lemma 4.10, $(\varphi_{\mathcal{U}}(S(e, u)), \varphi_{\mathcal{U}}(S(e, v)))$ is a not a good 2-separation of $M_{\mathcal{U}}$. However, $M_{\mathcal{U}}$ is simply the 2-sum of the torsos M_u and M_v along the element e. As T' is a decomposition tree, the torsos M_u and M_v have no good 2-separations. As T' is irrendundant, either one of them is 3-connected, or one is a circuit and the other is a cocircuit, by Lemma 2.3. In either case, $(\varphi_{\mathcal{U}}(S(e, u)), \varphi_{\mathcal{U}}(S(e, v)))$ is a good 2-separation of $M_{\mathcal{U}}$, a contradiction.

Suppose that there exists a good 2-separation $(S, S^{\complement})$ of M that is not an edge of T'. As $(S, S^{\complement})$ is nested with all the edges of T'. There must exist a vertex v of $T_{\mathcal{G}}$ such that $|\varphi_{\mathcal{U}}(S)|, |\varphi_{\mathcal{U}}(S^{\complement})| \geq 2$, $\varphi_{\mathcal{U}}(S) \cap \varphi_{\mathcal{U}}(S^{\complement}) = \emptyset$ where the localization $M_{\mathcal{U}}$ is equal to the torso M_v. By Lemma 4.9, $(\varphi_{\mathcal{U}}(S), \varphi_{\mathcal{U}}(S^{\complement}))$ is a good 2-separation of the torso M_v. However, as T' is a decomposition tree, the torso M_v has no good 2-separations, a contradiction. $\square$

This concludes the proof of Lemma 7.1.

8 The structure of the 2-separations of the dual

In this section, we prove Theorem 1.2. This essentially asserts that a tree-decomposition of a matroid is also a tree-decomposition of its dual. In particular, a matroid and its dual have the same unique irredundant tree-decomposition of uniform adhesion 2, and such that corresponding torsos are duals of one another. This theorem is implied by Proposition 8.1 and Claim 8.4 stated below. The former asserts that the k-separations of M and its dual coincide.

Proposition 8.1. *Let $k \geq 1$ be an integer. A k-separation of a matroid M is also a k-separation of its dual M^*.*

This is well known for finite matroids [8]. For infinite matroids this was established in [5, Lemma 18]. Here, we observe that Proposition 8.1 is a consequence of the following.

Claim 8.2. *Fix $S \subseteq E(M)$ and let $B \in \mathcal{B}(M)$, $B^* = B^{\complement} \in \mathcal{B}(M^*)$. Extend $B \cap S$ and $B^* \cap S^{\complement}$ to bases $B_S \in \mathcal{B}(M|S)$ and $B^*_{S^{\complement}} \in \mathcal{B}(M^*|S^{\complement})$, respectively. If $f = |B_S - (B \cap S)| < \infty$, then $|B^*_{S^{\complement}} - (B^* \cap S^{\complement})| = f$.*

To prove Claim 8.2 we use the following.

Lemma 8.3. [4, Lemma 3.7]
If $B, B' \in \mathcal{B}(M)$ satisfy $|B \setminus B'| < \infty$, then $|B \setminus B'| = |B' \setminus B|$.

Proof of Claim 8.2. Put $X = B_S \setminus (B \cap S)$ and $Y = B^*_{S^{\complement}} \setminus (B^* \cap S^{\complement}) = B^*_{S^{\complement}} \cap (B \cap S^{\complement})$. Noting that $M^*|S^{\complement} = M^* \setminus S = (M/S)^*$, we have $B^*_{S^{\complement}} \in \mathcal{B}((M/S)^*)$ so $(B \cap S^{\complement}) \setminus Y = S^{\complement} \setminus B^*_{S^{\complement}} = E((M/S)^*) \setminus B^*_(S^{\complement}) \in \mathcal{B}(M/S)$. By the definition of the contraction operation, it follows that $B' = B_S \cup (B \cap S^{\complement}) \setminus Y \in \mathcal{B}(M)$. As $B' \setminus B = X$ and $|X| = f < \infty$, then $f = |B' \setminus B| = |B \setminus B'| = |Y|$, by Lemma 8.3. ∎

Finally, we consider the torsos. For these we observe the following general property that holds for localizations (and not only for torsos). Let $M_{\mathcal{U}}$ be a localization of a connected matroid M at $\mathcal{U} = \{X_i : i \in I\}$ where $(X_i, X_i^{\complement})$ is a 2-separation of M for all i.

Claim 8.4. $(M^*)_{\mathcal{U}} = (M_{\mathcal{U}})^*$.

Proof. Let B be a base of M and S a 2-separation in M. It follows from Claim 8.2 that B is a base of $M|X_i$ if and only if $B^{\complement}$ is not a base of $M^*|X_i$.

Let $B_{\mathcal{U}}$ be a base of $M_{\mathcal{U}}$. There exists a base B_M of M such that $B_{\mathcal{U}} = (B_M \cap R(\mathcal{U})) \cup \{e_i : B_M \cap X_i \in \mathcal{B}(M|X_i)\}$. Now $B_M^{\complement}$ is a base of M^*. Let $B^*_{\mathcal{U}} = (B_M^{\complement} \cap R(\mathcal{U})) \cup \{e_i : B_M^{\complement} \cap X_i \in \mathcal{B}(M^*|X_i)\}$. Certainly, $B^*_{\mathcal{U}}$ is base of $(M^*)_{\mathcal{U}}$ and $B_{\mathcal{U}}^{\complement}$ is a base of $(M_{\mathcal{U}})^*$.

We claim that $B^*_{\mathcal{U}} = B^{\complement}_{\mathcal{U}}$. It is straightforward to see that $B^*_U \cap R(\mathcal{U}) = (B_{\mathcal{U}} \cap R(\mathcal{U}))^{\complement}$. So Let $v = e_i$ for some i. Now v is in $B_{\mathcal{U}}$ if and only if $B_M \cap X_i$ is a base of $M|X_i$. Similarly, v is in $B^*_{\mathcal{U}}$ if and only if $B^{\complement}_M \cap X_i$ is a base of $M^*|X_i$. As noted above, $B_M \cap X_i$ is a base of $M|X_i$ if and only if $B^{\complement}_M \cap X_i$ is not a base of $M^*|X_i$. Thus, $v \in B_{\mathcal{U}}$ if and only if $v \notin B^*_{\mathcal{U}}$ and the claim is proved. $\square$

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